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Convergence estimates of a projection-difference method for an operator-differential equation

Polina Vinogradova *

Department of Natural Sciences, Far Eastern State Transport University, 680021, Khabarovsk, Serisheva 47, Russia

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ABSTRACT

This article investigates the projection-difference method for a Cauchy problem for a linear operator-differential equation with a leading self-adjoint operator $A(t)$ and a subordinate linear operator $K(t)$ in Hilbert space. This method leads to the solution of a system of linear algebraic equations on each time level; moreover, the projection subspaces are linear spans of eigenvectors of an operator similar to $A(t)$. The convergence estimates are obtained. The application of the developed method for solving the initial boundary value problem is given.

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1. Introduction

In a number of works, for example, [1,2] the convergence of a Faedo–Galerkin method for non-stationary operator-differential equations is investigated in Banach space. Quite general conditions for existence and uniqueness of the solution of the Cauchy problem are established in the monographs [3–6]. It is known that the convergence velocity of approximate solutions much depends on the choice the basis functions. In [7] it is offered to choose the eigenvectors of the similar operator, which is time-independent and which forms the acute-angle with the leading operator of the equation, as basis functions. In [8,2] the concept of the weak solution of the operator-differential equation is introduced and the convergence of the approximate solutions by the projection method to the weak solution is established in the case of an arbitrary basis.

* Tel.: +7 4212 439518.
E-mail address: vpolina17@hotmail.com.

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In [9] projection procedures for operator-differential equation with the subordinated operators were investigated in the case of a special basis. Under specified conditions the convergence of approximate solutions in strong norm is established, and the convergence velocity depending on the degree of subordination is found.

Combined methods for solving the non-stationary equations were studied, for example, in [10–13], where error estimates in weak norms were obtained.

In this paper we study a projection-difference method for the Cauchy problem for a linear operator-differential equation with a leading self-adjoint operator $A(t)$ and subordinated to it, generally speaking, self-conjugate linear operator $K(t)$. It is supposed that the operators $A(t)$ and $K(t)$ are sufficiently smooth. The time discretization is based on a three-level difference scheme. While obtaining the asymptotic estimates the smoothness of the exact solution of the operator-differential equation plays an essential role. The increase of the smoothness of the exact solution of the Cauchy problem is a difficult task. In the present paper the way of getting the smooth approximate solutions by the Faedo–Galerkin method, depending on the smoothness of the operators is specified. The latter allows one to obtain the asymptotic estimates for the projection-difference method without the account of the smoothness of the exact solution. Easily checked restrictions on the operators and right-hand side of the equation are suggested.

2. Statement of the problem and auxiliary assertions

Let H_1 be separable Hilbert space densely embedded in a separable Hilbert space H. The norm in H will be denoted by $\|\cdot\|_H \equiv \|\cdot\|$. Let $\mathcal{B}_2 = \mathcal{B}_2(0, T; H)$ be a Hilbert space of all strongly measurable functions on the interval $[0, T]$ which range in H, and the norm

$$\|u(t)\|_{0,2} = \left(\int_0^T \|u(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

is finite. Consider the set of functions $u(t)$ ranging in H_1 and having continuous derivative in H. In set of such function we introduce the norm

$$\|u(t)\|_{1,2} = \left(\int_0^T (\|u'(t)\|^2 + \|u(t)\|_{H_1}^2)\mathrm{d}t\right)^{\frac{1}{2}}.$$

The completion of this set with respect to this norm is the Hilbert space $\mathcal{B}_2^1 = \mathcal{B}_2^1(0, T; H, H_1)$.

In the space H we consider the Cauchy problem

$$u'(t) + A(t)u(t) + K(t)u(t) = h(t), \qquad u(0) = 0. \tag{1}$$

In the sequel, we assume that the operators $A(t)$ and $K(t)$ satisfy the following conditions.

(1) $A(t)$ is self-adjoint positive definite operator in H with domain $D(A(t)) = H_1$.

(2) The operators $A(t)$ and $K(t)$ are three times strongly continuously differentiable on $[0, T]$ (see [5]); the derivatives $A^{(i)}(t), K^{(i)}(t)$ are determined on H_1 and range in H $(i = 1, 2, 3)$ and for any element $z \in H_1$ there exists a positive constant γ such that

$$\sup_{0\le t\le T} \|A^{(i)}(t)z\| \le \gamma \|z\|_{H_1},$$

$$\sup_{0\le t\le T} \|K^{(i)}(t)z\| \le \gamma \|z\|_{H_1}.$$

(3) There exists a constant $\beta \ge 0$ such that the inequality $(A'(t)z, z)_H \le \beta \|A^{\frac{1}{2}}(0)z\|^2$ holds for all $z \in H_1$.

(4) An operator B is similar to $A(0)$ (see [14]); i.e., B is self-adjoint positive definite operator with domain $D(B) = D(A(0))$. The operators $A(t)$ and B satisfy the acute-angle inequality (see [7]):

$$(A(t)z, Bz)_H \ge m\|A(0)z\|\|Bz\|,$$

where a constant $m > 0$ is independent of the choice of $z \in H_1$ and t.

(5) An operator $K(t)$ is subordinate to B with order α (see [5]); i.e., for each $z \in H_1$ there exists a positive constant M such that

$$\|K(t)z\| \le M\|Bz\|^{\alpha}\|z\|^{1-\alpha}, \quad 0 \le \alpha < 1. \tag{2}$$

We shall assume that B^{-1} and $A^{-1}(0)$ are compact in H. By $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ we denote the complete system of eigenvectors of B, so that $B\varphi_j = \lambda_j\varphi_j$, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \ldots$, and $\lambda_n \to \infty$ as $n \to \infty$. The linear span of the elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ will be denoted by H^n. Let P_n be the orthogonal projection in H onto H^n.

For problem (1), we consider the Faedo–Galerkin method

$$u_n'(t) + P_nA(t)u_n(t) + P_nK(t)u_n(t) = P_nh(t), \quad u_n(0) = 0. \tag{3}$$