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Error analysis and applications of the Fourier–Galerkin Runge–Kutta schemes for high-order stiff PDEs

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ABSTRACT

An integrating factor mixed with Runge-Kutta technique is a time integration method that can be efficiently combined with spatial spectral approximations to provide a very high resolution to the smooth solutions of some linear and nonlinear partial differential equations. In this paper, the novel hybrid Fourier–Galerkin Runge–Kutta scheme, with the aid of an integrating factor, is proposed to solve nonlinear high-order stiff PDEs. Error analysis and properties of the scheme are provided. Application to the approximate solution of the nonlinear stiff Korteweg–de Vries (the 3rd order PDE, dispersive equation), Kuramoto–Sivashinsky (the 4th order PDE, dissipative equation) and Kawahara (the 5th order PDE) equations are presented. Comparisons are made between this proposed scheme and the competing method given by Kassam and Trefethen. It is found that for KdV, KS and Kawahara equations, the proposed method is the best.

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1. Introduction

Hybrid schemes based on a combination of the spectral and discrete variable methods have been used in recent years for approximating solutions for a variety of stiff nonlinear PDEs [1–3]. Numerical simulation of the stiff nonlinear PDEs is not an easy task if accurate solutions are to be found efficiently (for example see [4–10]). More recently, Kassam and Trefethen attempt to solve the Kuramoto–Sivashinsky (KS) and Korteweg–de Vries (KdV) equations [3] efficiently by modifying the method introduced with Cox and Matthews [2].

The KdV equation in one dimension can be derived based on nonlinear waves equations in shallow water. It has been found in other physical models such as ion acoustic waves in a plasma [11] and acoustic waves in an anharmonic crystal [12]. For more details one can see the work of Drazin and Johnson [13] and Infeld and Rowlands [11].

The one dimensional KS equation has been studied in the context of inertial manifolds and finite-dimensional attractors, and in the numerical simulations of dynamical behaviors [14]. On the other hand the KS equation models the effect of the particles being knocked out of the interface by the bombarding ions [15].

The Kawahara equation [16,17], is the fifth-order dispersive-type partial differential equation describing one-dimensional propagation of small-amplitude long waves in various problems of fluid dynamics and plasma physics [18,19]. The Kawahara equation is also known as fifth-order KdV or the special version of the Benney–Lin equation [20,21].

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Nomenclature

$\mathcal{L}$	Linear operator (continuous)
$\mathcal{N}$	Nonlinear operator (continuous)
$\mathbb{L}$	Discretized linear operator
$\mathbb{N}$	Discretized nonlinear operator
L	Discretized Fourier–Galerkin linear operator
x	coordinate in horizontal direction
t	time
$f(x,t)$	Real valued source function
$u(x,t)$	Continuous solution
$u^N(x,t)$	Truncated Fourier approximate solution
u_t	Derivative of u with respect to t
u_x	Derivative of u with respect to x
$\hat{u}(t)$	Fourier coefficients of $u(x,t)$
$\hat{u}_k(t)$	kth Fourier coefficients in Fourier–Galerkin method
$\hat{f}_k(t)$	kth Fourier coefficients of $f(x,t)$

Stiff systems of ordinary differential equations arise commonly when solving time dependent partial differential equations by spectral methods, and their numerical solution requires spatial treatment if accurate solutions are to be found efficiently. On the other hand, many time dependent partial differential equations contain low-order nonlinear terms with high-order linear terms. Examples include the third order KdV, the fourth order KS and the fifth order Kawahara equations. In this paper we propose and test the accuracy of a class of numerical methods for integrating stiff systems based on high-order approximations in space and time. The aim of this paper is to bring together three schemes in which the Fourier Galerkin scheme in space and the mixture of integrating factor with Runge–Kutta technique in time (FGIFRK4) is used to discretize the model problems. When we solve a time dependent PDE, it is natural to write the solution as a sum of Fourier modes with time dependent coefficients. The corresponding spectral methods have been shown to be successful for a wide range of applications [22–24]. In problems with periodic boundary conditions, a basis such as Fourier modes are appropriate since the linearized system is diagonal.

The structure of this paper is as follows. In Section 2 we describe the proposed novel scheme. In the third section we discuss the truncation error for the FGIFRK4 scheme. Section 4 provides the stability analysis. In Section 5 we summarize the results of comparison between the Fourier Galerkin exponential time differencing method with fourth order Runge–Kutta time stepping (FGETDRK4), Fourier Collocation exponential time differencing method with fourth order Runge–Kutta time stepping (FCETDRK4) and FGIFRK4 for the KdV, KS and Kawahara equations. Section 6 concludes the paper, and summarizes the advantages of the FGIFRK4 scheme.

2. Novel scheme: FGIFRK4

Consider the following stiff nonlinear PDE

$$u_t = \mathcal{L}u + \mathcal{N}(u,t) + f(x,t), \quad x \in [0, 2\pi], t > 0, \tag{1}$$

with periodic boundary conditions and one initial condition, where $f(x,t)$ is a given real valued function of x and t, $\mathcal{L}$ is the linear operator of arbitrary order and $\mathcal{N}$ is the nonlinear operator in the form

$$\mathcal{N}(u,t) = u_x u. \tag{2}$$

Discretization of (1) in space gives rise to a system of ODEs,

$$u_t = \mathbb{L}u + \mathbb{N}(u,t) + f_N(t), \quad t > 0, \tag{3}$$

with initial condition

$$u(x, t=0) = u_0(x), \tag{4}$$

where f_N contains the prescribed right hand side. In the following theorem we provide the Fourier coefficients of the approximate solution to problem (1) in a fixed spatial mesh.

Theorem 1. *Assume that Eq. (1) is discretized by the Fourier Galerkin technique in space and the integrating factor with fourth order Runge–Kutta in time (FGIFRK4), where the boundary conditions are periodic. Then the Fourier coefficients are:*

$$\hat{u}(t_{n+1}) = E_2\hat{u}(t_n) + \frac{1}{6}(E_2 a_1 + 2E_1(b_1 + c_1) + d), \tag{5}$$

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