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Triple positive solutions to third order three-point BVP with increasing homeomorphism and positive homomorphism☆

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ABSTRACT

In this paper, we investigate the existence of triple positive solutions for nonlinear differential equations boundary value problems with increasing homeomorphism and positive homomorphism operator. By using fixed point theorems in cones, we establish results on the existence of three positive solutions with suitable growth conditions imposed on the nonlinear term. As applications, two examples are given to demonstrate our result. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

In this paper, we consider the following third order three-point boundary value problem

$$\begin{cases}(\phi(u''(t)))' + a(t)f(u(t)) = 0, \ 0 < t < 1,\\ u(0) = \beta u(\xi), \qquad u'(1) = 0, \qquad \phi(u''(0)) = \delta\phi(u''(\xi)),\end{cases} \tag{1.1}$$

here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$, f, a, β, δ, ξ satisfy:

(H_1) $0 < \xi < 1,\ 0 < \beta < 1,\ 0 < \delta < 1$;
(H_2) $f : [0, \infty) \to \mathbb{R}^+$ is continuous, $a \in C([0, 1], \mathbb{R}^+)$ and there exists $t_0 \in [0, 1]$ such that $a(t_0) > 0$, where $\mathbb{R}^+ = [0, \infty)$.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and a positive homomorphism, if the following conditions are satisfied:

(1) if $x \le y$, then $\phi(x) \le \phi(y)$ for all $x, y \in \mathbb{R}$;
(2) ϕ is a continuous bijection and its inverse mapping ϕ^{-1} is also continuous;
(3) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in [0, +\infty)$.
In the above definition, we can replace condition (3) by the following stronger condition:
(4) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

Remark 1.1. If conditions (1), (2) and (4) hold, then it implies that ϕ is homogeneous, generating a p-Laplacian operator, i.e., $\phi(x) = |x|^{p-2}x$, for some $p > 1$.

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Third order differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, the existence and multiplicity of positive solutions for nonlinear third order ordinary differential equations with a three-point boundary value problem (BVP for short) have been studied by several authors. An interest in triple solutions evolved from the Leggett–Williams multiple fixed points theorem [1]. And lately, two triple-fixed-point theorems due to Avery [2] and Avery and Peterson [3], have been applied to obtain triple solutions for certain three point boundary value problems of third order ordinary differential equations. For example, one may see [4–10] and references therein. In [5], D. R. Anderson considered the the following third order nonlinear boundary value problem

$$\begin{cases} x'''(t) = f(t, x(t)), \quad t_1 < t < t_3, \\ x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \end{cases}$$

He used the Krasnoselskii and Leggett–Williams fixed point theorems to prove the existence of solutions to the nonlinear boundary value problem. In [9], Sun considered the the following third order nonlinear boundary value problem

$$\begin{cases} u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \qquad u''(1) = 0. \end{cases}$$

He used the fixed point theorems due to Avery and Peterson to establish results on the existence of positive solutions to the nonlinear boundary value problem.

On the other hand, the boundary value problems with a p-Laplacian operator have also been discussed extensively in the literature, for example, see [11–17,10]. In [10], Zhou and Ma studied the existence of positive solutions for the following third order generalized right-focal boundary value problem with a p-Laplacian operator

$$\begin{cases} (\phi_p(u''))'(t) = q(t)f(t, u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $1 < p \le 2$. They established a corresponding iterative scheme for the boundary value problem by using the monotone iterative technique.

However, to the best of our knowledge, for the increasing homeomorphism and positive homomorphism operator the research has proceeded slowly. In [18], Liu and Zhang studied the existence of positive solutions of quasilinear differential equation

$$\begin{cases} (\phi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1, \\ x(0) - \beta x'(0) = 0, \qquad x(1) + \delta x'(1) = 0, \end{cases}$$

here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. They obtained the existence of one or two positive solutions by using a fixed-point theorem in cones. For other results which involved an increasing homeomorphism and positive homomorphism operator, the readers are referred to [19–24].

However, there are not many results concerning the existence of triple positive solutions to the third order three-point boundary value problems of nonlinear differential equations with increasing homeomorphism and positive homomorphism operator so far. Whether or not we can obtain triple positive solutions to these kinds of boundary value problems still remains unknown. Motivated greatly by the results mentioned above, especially reference [9], in this paper, we will consider the existence of positive solutions(at least three) to BVP (1.1) by using fixed-point theorems in cones. We improve and generate a p-Laplacian operator and establish some criteria for the existence of triple positive solutions to BVP (1.1).

The methods used in our work will depend on applications of a fixed point theorem due to Avery–Peterson [3] which deals with fixed points of a cone-preserving operator defined on an ordered Banach space, and another fixed point theorem which can be found in [25]. The emphasis here is the differential equation with increasing homeomorphism and the positive homomorphism operator.

The paper is planned as follows. In Section 2, for convenience of the readers we give some definitions and lemmas in order to prove our main results. Section 3 is developed to present and prove our main results. As applications, two examples are given to demonstrate our results in Section 4.

2. Preliminaries and lemmas

In this section, we provide some background materials cited from the cone theory in Banach spaces, and we then state two triple fixed points theorem for a cone preserving operator. The following definitions and lemmas can be found in the monograph by Deimling [26] as well as the monograph by Guo and Lakshmikanthan [25].

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.