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On an iterative algorithm with superquadratic convergence for solving nonlinear operator equations

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ABSTRACT

We study an iterative method with order $(1+\sqrt{2})$ for solving nonlinear operator equations in Banach spaces. Algorithms for specific operator equations are built up. We present the received new results of the local and semilocal convergence, in case when the first-order divided differences of a nonlinear operator are Hölder continuous. Moreover a quadratic nonlinear majorant for a nonlinear operator, according to the conditions laid upon it, is built. A priori and a posteriori estimations of the method's error are received. The method needs almost the same number of computations as the classical Secant method, but has a higher order of convergence. We apply our results to the numerical solving of a nonlinear boundary value problem of second-order and to the systems of nonlinear equations of large dimension.

1. Introduction

Let F be a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Let us consider for solving the equation

$$F(x) = 0 \tag{1.1}$$

the algorithm

$$\begin{aligned} x_{n+1} &= x_n - [\delta F(x_n, y_n)]^{-1} F(x_n), \\ y_{n+1} &= x_{n+1} - [\delta F(x_n, y_n)]^{-1} F(x_{n+1}), \quad n = 0, 1, \ldots, \end{aligned} \tag{1.2}$$

where, for each $x_n, y_n \in D$, $\delta F(x_n, y_n)$ is bounded linear operator from X to Y; x_0, y_0 are given.

The history of this method is rather rich. For function $F : \mathbb{R} \to \mathbb{R}$ method (1.2) was first investigated in the work [1] as a Secant method. For Banach spaces a differential analog of this method

$$\begin{aligned} x_{n+1} &= x_n - [F'(y_n)]^{-1} F(x_n), \\ y_{n+1} &= x_{n+1} - \frac{1}{2}[F'(y_n)]^{-1} F(x_{n+1}), \quad n = 0, 1, \ldots, x_0 = y_0 \end{aligned} \tag{1.3}$$

was proposed in [2]. Later independently in different forms and under different conditions it was also proposed and investigated in the works of the other authors [3–5]. Rather deep investigations of the methods (1.2) and (1.3) were

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conducted in the works of Ukrainian mathematicians M.Ya. Bartish and Yu.M. Shcherbyna [2,6–8]. However while exploring the method (1.2) the existence of divided difference of the second-order of operator F as well as bounded norm or Lipschitz (Hölder) continuity of divided difference of the second-order were required.

In this work we define divided differences for some nonlinear operators and apply algorithm (1.2) for solving concrete types of operator equations.

We conduct an investigation of the local convergence of the method (1.2) (Cauchy type conditions) and the semilocal convergence (Kantorovich type conditions [9]) under much weaker conditions, than in all the other known works. Particularly we demand only the existence and continuity by Hölder, just as for the classical Secant method [10–12]. The formula of dependance of the order of convergence on the Hölder constant has been received. Moreover we use for the first time the majorant methodology for investigating the semilocal convergence of the method (1.2). We have proved the theorem about uniqueness of the solution.

Investigated in [13,14] iterative difference methods also demand the existence and continuity by Lipschitz of the second-order divided differences. They converge locally to a solution with order of convergence 1.839… and 2 respectively. For rather smooth functions F and simple zeros we can show that the approximations created by the iterative method (1.2) converge asymptotically to the solution at least with order of convergence $1+\sqrt{2}$. The number of calculations on each iteration is practically the same as in a classical Secant method.

In this work we will consider an open convex subset D of the space X and suppose that F is differentiable by Frechet in in D. We suppose that the divided difference $\delta F(x,y)$ satisfies the Hölder conditions, if there exists a nonnegative constant k such that

$$\|\delta F(x,y)-\delta F(u,v)\| \le k(\|x-u\|^{\alpha}+\|y-v\|^{\alpha}), \quad \alpha\in(0,1] \tag{1.4}$$

for all $x, y, u, v \in D$ with $x \neq y$ and $u \neq v$. In this case we will state that F has on D a continuous by Hölder divided difference. With that as we know [10], exists a Frechet derivative of F in D and it satisfies $\delta F(x,x) = F'(x),\ x \in D$.

The paper is organized as follows: In Section 2, we give a definition of the first-order divided differences for specific nonlinear operators and write down iterative algorithm (1.2) for corresponding types of nonlinear operator equations. In Sections 3–5 we give local and semilocal convergence theorems, the estimates of radii for the convergence ball of method (1.2), the uniqueness of the solution, a posteriori estimation of the error of the method (1.2). In Section 6, we present some numerical experiments.

2. Divided differences and algorithms of method (1.2) for specific operator equations

Definition 2.1. Let F be a nonlinear operator defined on a subset D of a linear space X with values in linear space Y and let x, y be two points of D. A linear operator from X into Y, denoted $\delta F(x,y)$, which satisfies the condition

$$\delta F(x,y)(x-y) = F(x)-F(y) \tag{2.1}$$

is called a divided difference of F at the points x and y.

Let us consider a presentation of the first-order divided differences for specific nonlinear operators and write down iterative algorithm (1.2) for corresponding types of nonlinear operator equations. Let us note that an overview of different definitions of divided differences and relations between them is provided in [15].

1. Now consider the case where $F:\mathbb{R}^n\to\mathbb{R}^n$. Let us apply the method (1.2) to solving a nonlinear system of n real equations with n variables

$$F_i(x_1,\ldots,x_n)=0, \quad i=1,2,\ldots,n. \tag{2.2}$$

In this case the divided difference $\delta F(x,y)$ is represented by the matrix with entries [10,13,16]

$$\delta F(x,y)_{ij} = \frac{F_i(x_1,\ldots,x_j,y_{j+1},\ldots,y_n)-F_i(x_1,\ldots,x_{j-1},y_j,\ldots,y_n)}{x_j-y_j}, \quad i,j=1,2,\ldots,n, \tag{2.3}$$

where function $\delta F(x,y)_{ij}$ with $x_j=y_j$ equals $\frac{\partial F_i}{\partial x_j}(x_1,\ldots,x_j,y_{j+1},\ldots,y_n)$.

Successive approximations $x^{(k+1)}, y^{(k+1)}$ to solution x^* in case (2.2) are determined from the system of equations for corrections

$$\begin{aligned}
&\sum_{j=1}^{n} H_{ij}^{(k)}(x_j^{(k+1)}-x_j^{(k)}) = -F_i(x_1^{(k)},\ldots,x_n^{(k)}),\\
&\sum_{j=1}^{n} H_{ij}^{(k)}(y_j^{(k+1)}-x_j^{(k+1)}) = -F_i(x_1^{(k+1)},\ldots,x_n^{(k+1)}),\\
&H_{ij}^{(k)} = \frac{F_i(x_1^{(k)},\ldots,x_j^{(k)},y_{j+1}^{(k)},\ldots,y_n^{(k)})-F_i(x_1^{(k)},\ldots,x_{j-1}^{(k)},y_j^{(k)},\ldots,y_n^{(k)})}{x_j^{(k)}-y_j^{(k)}}, \quad i=1,2,\ldots,n;\ k=0,1,2,\ldots.
\end{aligned} \tag{2.4}$$