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Application of modified homotopy perturbation method for solving the augmented systems☆

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ABSTRACT

In this paper, a new approach is proposed for solving the augmented systems. Based on the modified homotopy perturbation method, we construct the new iterative methods and derive the sufficient and necessary conditions for guaranteeing its convergence. Some numerical experiments show that this method is more simple and effective.

1. Introduction

The augmented system is of the form

$$\begin{pmatrix} A & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric and positive definite matrix, $B \in R^{m\times n}$ $(m \geq n)$ is a matrix of full column rank, and B^{T} is the transpose of matrix B, $b \in R^m$ and $q \in R^n$ are two given vectors. This class of problems appears in many different fields of the scientific computing and engineering applications, such as the constrained optimization [1–3], the finite element method or the finite volume method for solving the Navier–Stokes equations [4–8], and the constrained least squares problems and the generalized least squares problems [9,10], etc. There have been a great deal of iterative methods for solving the augmented system (1.1). Among them, the preconditioned iterative methods were provided firstly by Santos and co-works in [11]. Several variants of the SOR method and the preconditioned conjugate gradient methods were proposed for solving the general augmented system (1.1) arising from the generalized least squares problems by Yuan

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and co-workers in [9,10]. The preconditioned MINRES method, the QMR method, the preconditioned GMRES method, the SOR-like methods and the generalized SOR-like methods were investigated respectively for solving the augmented system arising from finite element approximations to the Stokes equations in [12,8,1,13–15]. Recently, an iterative method with variable relaxation parameters [16,17], the generalized successive overrelaxation methods [18,19,10], the parameterized inexact Uzawa methods [20–23] and the fast Uzawa algorithms [24,25] were studied for solving the augmented systems and the generalized saddle point problems, respectively.

In this paper, we consider a new approach for solving the augmented system (1.1). Since the second diagonal block matrix is null, we introduce a full-rank matrix Q with small parameter $p \in [0, 1]$ and construct the new iterative methods by using the modified homotopy perturbation method [26]. The sufficient and necessary conditions for guaranteeing its convergence are derived. Four kinds of perturbation cases of the new approach are studied respectively. Finally, four special choices of the full-rank matrix Q are considered for solving problem (1.1). Numerical experiments show that this method is more simple and effective.

The outline of this paper is as follows. In Section 2, we replace the null block in problem (1.1) by some nonsingular matrix $(1-p)\alpha Q$ so that we can apply the modified homotopy perturbation method to the small parameter p. Moreover, we study the convergence of the new iterative methods for four special cases. In Section 3, we make four special choices for Q and give some numerical experiments for our algorithms. The numerical experiments show that our methods work well for problem (1.1) arising from real problems. Finally, the conclusions are made in Section 4.

2. The construction of iterative methods

In order to find the solution of problem (1.1), we choose the following four different auxiliary systems.

2.1. Case one

The first auxiliary system is as follows

$$\begin{pmatrix} A & B \\ B^{\mathrm{T}} & (1-p)\alpha Q \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{2.1}$$

where $\alpha \neq 0$ is the accelerating parameter, $Q \in R^{n\times n}$ is a given matrix and needs to be non-singular and "easy" to invert, $p \in [0, 1]$ is an imbedding parameter. Hence, it is obvious that when $p = 1$, problem (1.1) is a degenerated form of problem (2.1).

The changing process of p from 0 to 1 is just that of the solution of problem (2.1) from the solution of problem (1.1). In topology, this is called deformation. Applying the homotopy perturbation technique [27], due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of problem (2.1) can be expressed as a series in p

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + p \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + p^2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \cdots + p^n \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \cdots, \tag{2.2}$$

when $p \to 1$, problem (2.1) corresponds to problem (1.1), and solution (2.2) becomes the approximate solution of problem (1.1), namely

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{p\to 1} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \cdots + \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \cdots. \tag{2.3}$$

Substituting (2.2) into problem (2.1), and equating the coefficients of like powers of p, we obtain the following systems

$$\begin{aligned}
&p^0 : \begin{pmatrix} A & B \\ B^{\mathrm{T}} & \alpha Q \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \\
&p^1 : \begin{pmatrix} A & B \\ B^{\mathrm{T}} & \alpha Q \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha Q y_0 \end{pmatrix}, \\
&p^2 : \begin{pmatrix} A & B \\ B^{\mathrm{T}} & \alpha Q \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha Q y_1 \end{pmatrix}, \\
&\vdots \\
&p^k : \begin{pmatrix} A & B \\ B^{\mathrm{T}} & \alpha Q \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha Q y_{k-1} \end{pmatrix}, \\
&\vdots
\end{aligned} \tag{2.4}$$

From (2.4), we can see that if $\alpha Q - B^{\mathrm{T}}A^{-1}B$ is a nonsingular matrix, then $(x_0, y_0)^{\mathrm{T}} \sim (x_k, y_k)^{\mathrm{T}}$ can be solved respectively