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Full rank interpolatory subdivision schemes: Kronecker, filters and multiresolution

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ABSTRACT

In this extension of earlier work, we point out several ways how a multiresolution analysis can be derived from a finitely supported interpolatory matrix mask which has a positive definite symbol on the unit circle except at -1. A major tool in this investigation will be subdivision schemes that are obtained by using convolution or correlation operations based on replacing the usual matrix multiplications by Kronecker products.

1. Introduction

In the recent papers [1,2] we introduced and studied *full rank interpolatory* vector subdivision schemes. In particular, we investigated in [1] an extension of positivity of the symbol to the vector case and the implied convergence of associated subdivision schemes. Keeping in mind that the *symbol* of a finitely supported *mask* $\mathcal{A} = (\boldsymbol{A}_j : j \in \mathbb{Z})$ is the matrix valued Laurent polynomial $\boldsymbol{A}(z) = \sum_j \boldsymbol{A}_j z^j$, one of the main results in [1] can be formulated as follows.

Theorem 1. *If $\mathcal{A}$ is a finitely supported mask such that the associated symbol $\boldsymbol{A}(z)$ satisfies $\boldsymbol{A}(-1) = \boldsymbol{0}$, the interpolatory condition $\boldsymbol{A}(z) + \boldsymbol{A}(-z) = 2\boldsymbol{I}$, $z \in \mathbb{C}^*$ and is positive definite on $\{z \in \mathbb{C} : |z| = 1\} \setminus \{-1\}$, then there exists a canonical spectral factor $\mathcal{B}$ of $\mathcal{A}$ such that $\boldsymbol{A}(z) = \frac{1}{2}\boldsymbol{B}^H(z)\,\boldsymbol{B}(z)$ and an orthogonal $\mathcal{B}$-refinable function $\boldsymbol{G} \in L_2^{r\times r}(\mathbb{R})$.*

This result allowed us to introduce and investigate various different subdivision schemes which were in part classical stationary ones, but there also naturally appeared a nonstandard type of subdivision scheme which we called *correlated* since it consists of applying the subdivision scheme to the data sequence as well as to the "identity sequence" $\boldsymbol{I}\delta$ and then correlating the results:

$$\mathcal{S}_{\boldsymbol{B}}^n := \frac{1}{2^n}\left(S_{\mathcal{B}}^n \delta \boldsymbol{I}\right)^T \star S_{\mathcal{B}}^n, \tag{1}$$

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where $S_{\mathcal{B}}$ denotes the usual stationary subdivision scheme with respect to $\mathcal{B}$. The notational details will be explained in the next section.

Each of these subdivision schemes – if convergent – defines a *refinable function* where refinability has to be understood either in the classical or in a more general sense. These schemes are:

(1) the subdivision scheme $S_{\mathcal{A}}$ itself, based on the full rank interpolatory mask $\mathcal{A}$ whose symbol $\boldsymbol{A}(z)$ is assumed to be positive definite on the unit circle. The associated matrix refinable function (if it exists) is a cardinal function $\boldsymbol{F}$ and a partition of the identity, that is, $\boldsymbol{F}(k) = \delta_{k,0}\,\boldsymbol{I}$ and $\sum_k \boldsymbol{F}(\cdot - k) = \boldsymbol{I}$. However, the convergence of $S_{\boldsymbol{A}}$ or, equivalently, the existence of the cardinal refinable function $\boldsymbol{F}$ could not be concluded from the assumptions of Theorem 1. Whether or not this refinable function exists is still an open question.

(2) The subdivision scheme $S_{\mathcal{B}}$ based on the full rank mask $\mathcal{B}$ whose symbol is the canonical spectral factor, of $\boldsymbol{A}(z)$, i.e.

$$\boldsymbol{A}(z) = \frac{1}{2}\boldsymbol{B}^H(z)\boldsymbol{B}(z).$$

According to [1], the associated matrix refinable function $\boldsymbol{G}$ exists, is of full rank and is orthogonal so that the associated subdivision scheme converges in $L_2^r(\mathbb{R})$.

(3) The *correlated subdivision scheme* $\mathcal{S}_{\mathcal{B}}$ based on the full rank mask $\mathcal{B}$. The associated limit matrix function $\boldsymbol{F}_\star \in C_u^{r\times r}(\mathbb{R})$, whose existence was proved in [1], is refinable in the following sense:

$$\boldsymbol{F}_\star = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \boldsymbol{B}_k^T \boldsymbol{F}_\star(2\cdot - j + k)\boldsymbol{B}_j. \tag{2}$$

Furthermore, $\boldsymbol{F}_\star$ is cardinal, $\boldsymbol{F}_\star(k) = \delta_{0k}\,\boldsymbol{I}$, and satisfies the partition of the identity property

$$\sum_{k\in\mathbb{Z}} \boldsymbol{F}_\star\,(\cdot - k) = \boldsymbol{I}.$$

Note that in the scalar case $r = 1$ the two functions $\boldsymbol{F}$ and $\boldsymbol{F}_*$ coincide as also do the respective subdivision schemes and refinement equations.

(4) The subdivision scheme $S_{\mathcal{C}}$ based on the mask $\mathcal{C}$ defined by means of the Kronecker product of symbols as

$$\boldsymbol{C}(z) = \frac{1}{2}\boldsymbol{B}(z) \otimes \boldsymbol{B}\left(z^{-1}\right), \tag{3}$$

and its associated *vector* refinable function Φ, that is, a solution of the refinement equation

$$\Phi = \sum_{k\in\mathbb{Z}} \boldsymbol{C}_k^T \Phi\,(2\cdot - k)$$

which could be derived directly from $\boldsymbol{F}_\star$.

In this paper, we will consider two more aspects. First, we will investigate more closely properties of the subdivision scheme $S_{\mathcal{C}}$ based on the full rank mask $\mathcal{C}$, proving that the subdivision scheme converges and that its associated full rank basic limit function $\boldsymbol{H}$ is stable and thus can be used to define a multiresolution analysis (usually abbreviated as "MRA"). Second, we will define multiresolution analyses and/or filter banks associated to all the above-mentioned "refinable" functions, pointing out some of the connections between them. All these MRAs will be suitable for *vector data processing*, and could be applied to vector valued time series, for example, in the analysis of EEG signals, cf. [3].

2. Notation and background

For $r \in \mathbb{N}$ we write an $r \times r$ matrix $\boldsymbol{A} \in \mathbb{R}^{r\times r}$ as $\boldsymbol{A} = \left[A_{jk} \ : \ j, k = 1, \ldots, r\right]$ and denote by $\ell_\infty^{r\times r}(\mathbb{Z})$ the Banach space of all $r \times r$ matrix valued bi-infinite sequences with bounded operator norm, considered as convolution operators on $\ell^{r\times 1}(\mathbb{Z})$. More precisely, $\mathcal{A} = \left(\boldsymbol{A}_j \ : \ j \in \mathbb{Z}\right) \in \ell_\infty^{r\times r}(\mathbb{Z})$, is defined by

$$\|\mathcal{A}\| := \|\mathcal{A}\|_\infty := \sum_{j\in\mathbb{Z}} \left|\boldsymbol{A}_j\right|_\infty < \infty, \qquad |\boldsymbol{A}|_\infty = \max_{1\le j\le r} \sum_{k=1}^r \left|A_{jk}\right|. \tag{4}$$

For notational simplicity we write $\ell_\infty^r(\mathbb{Z})$ for $\ell_\infty^{r\times 1}(\mathbb{Z})$ and denote vector sequences by lowercase letters. Moreover, $C_u^{r\times r}(\mathbb{R})$ will denote the Banach space of all uniformly continuous uniformly bounded $r \times r$ matrix valued functions on $\mathbb{R}$ with the norm

$$\|\boldsymbol{F}\|_\infty := \sup_{x\in\mathbb{R}} |\boldsymbol{F}(x)|_\infty < \infty.$$

For two matrix sequences we introduce the *convolution* "$*$" and the *correlation* "$\star$" defined, respectively, as

$$(\mathcal{A} * \mathcal{B})_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j-k}\,\boldsymbol{B}_k, \qquad (\mathcal{A} \star \mathcal{B})_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j+k}\,\boldsymbol{B}_k$$