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Curve and surface construction using Hermite subdivision schemes

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ABSTRACT

In this paper we present a very efficient Hermite subdivision scheme, based on rational functions, and outline its potential applications, with special emphasis on the construction of cubic-like B-splines — well suited for the design of constrained curves and surfaces.

1. Introduction

Due to their flexibility, their simple implementation and (often) their low computational cost, subdivision schemes are a very popular tool for representation and manipulation of curves and surfaces. In the last two decades, several interpolatory and non-interpolatory subdivision schemes have been proposed by many authors (see e.g. the survey paper [1]). More recently, the so-called *Hermite C^1 interpolatory schemes* (HC^1) have been analyzed. Among them we recall the relevant subset of *Merrien schemes*, [2–12]. Perhaps, the main appeal of the HC^1 schemes (as well as any Hermite scheme) lies in their localness. Indeed, in the commonly accepted analytic interpretation, they are seen as the evaluation of a *Hermite interpolant*, taken from some suitable 4D space, say $\mathcal{P}$, and of its derivative at a given point, usually the mid point.[1] Many and well known are the advantages of local versus global formulas: here we limit ourselves to put in evidence the two more significant ones for this paper. The first concerns the construction of C^1 piecewise curves or surfaces, which is trivial provided that suitable Hermite conditions are assigned at the common points of the 1D or the 2D grid. The second concerns a possible easier local control of the shape. Indeed, if from one side it is now accepted that any efficient tool for design, interpolation, approximation etc. must give the user some control on the form of the curve or surface, on the other it is well known to any researcher working in this field how difficult it is to manage *global shape constraints*. The papers on the HC^1 schemes cited in the references are all devoted to detailing these aspects.

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1 Such an interpretation does not imply in general that the limit function produced by the subdivision scheme belongs to $\mathcal{P}$. This is of course the case when $\mathcal{P}$ is the space of cubic polynomials, but not for other common choices (rational functions, exponential functions, etc.) at least in the simplest stationary and uniform version of the HC^1 scheme.

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Let us turn our attention to the efficiency and smoothness of HC^1 schemes. The efficiency claimed at the beginning of this section is completely achieved when *stationary and uniform* schemes are used. The evaluation of a cubic polynomial at mid point via the *de Casteljau* algorithm gives the most famous example (note, in passing that the de Casteljau algorithm gives also the derivative at mid point [13] and thus it can be viewed as the simplest implementation of the simplest HC^1 scheme). Stationary and uniform HC^1 schemes also allow imposition of shape constraints (see [10,6,7] and references therein) but their simplicity has as a counterpart a low degree of smoothness of the limit, which is of course C^1, but typically exhibits a "fractal" behavior in the graph of the first derivative. To obtain a higher smoothness, still retaining shape preservation, requires *non-uniform* and/or *non-stationary* approaches, which is often quite costly.

The *geometric construction* proposed in [14] goes in this direction. However, surprisingly and fortunately, in the effective implementation of the theoretical results of [14], using the Hermite interpolant from a suitable subspace of rational functions – which will be referred to as the *rational Hermite scheme* – we obtained a scheme with the following properties:

- at any level of subdivision the scheme can be described in terms of a piecewise rational function (actually piecewise cubic except in the first and in the last subinterval) of class C^2 in the given (closed) interval;

- the scheme converges to a C^2 function in the open interval;

- each subdivision step can be described by three bidiagonal and totally positive matrices applied to the "generalized" Bézier control points of the above mentioned rational function;

- the scheme coincides with the de Casteljau one with the only exception of the first and the last subinterval for each level, or in other words, is stationary and uniform "almost everywhere".

The positive consequences of these properties are evident. The piecewise function representation cited in the first item allows us to use the general results of [15] and [16] and construct both a Bernstein–Bézier representation (for a single interval) and a classical C^2 B-spline representation (when more intervals are connected in the construction of curves and surfaces). Second, now C^2 continuity and shape preservation can be achieved simultaneously, in contrast to the usual C^1 reached with HC^1 shape preserving schemes. Third, since the bidiagonal and totally positive matrices describe corner cutting steps, the scheme is *inherently shape preserving* in the sense that the shape of the initial control polygon is maintained through the subdivisions. In other words, we do not have to worry about the set-up of level and interval dependent shape constraints. Clearly, the advantages are overwhelming in the construction of shape preserving surfaces. Finally, the computational cost is the lowest possible, being equivalent to that of the cubic de Casteljau algorithm. Again, the advantages are more important for surface construction.

Summarizing, with the *rational Hermite scheme* we have at our disposal a tool which provides a mathematical object with shape preserving properties and high smoothness at the same computational cost and equivalent to cubic polynomials. In this paper we exploit some useful potential applications of this powerful scheme including the description of new 4D spaces possessing shape parameters and the construction of the corresponding B-splines. Due to space limitations, we have not gone deep into other possible interesting theoretical aspects but concentrated on an outline of the main applications. For the same reason, we have provided a very limited number of numerical and graphical examples.

The paper is divided into six sections, which essentially reflect the above comments. In the next one, after recalling some basic material from the references in order to make the paper self-contained, we present a class of interesting 4D spaces of piecewise rational functions, possessing shape parameters, defined by means of (some steps of) the subdivision scheme and we describe the construction of *generalized* Bernstein bases for them. In Section 3 we describe the geometric construction of the corresponding *generalized* B-spline basis, both in the 1D and in the 2D (tensor-product) case. Sections 4 and 5 are devoted to bivariate extensions of the considered scheme. The construction of shape preserving composite C^1 tensor-product surfaces, interpolating Hermite data at grid points, is presented in Section 4 while Section 5 is dedicated to the equally important construction of C^1 Boolean sum surfaces, which can be used to interpolate a network of *arbitrary given curves*. Finally, Section 6 contains final comments and remarks.

2. Rational Hermite subdivision schemes and related spaces

In this section we present the basic concepts related to the rational Hermite subdivision scheme (*rH* for short) and to suitable spaces of functions which can be constructed from it. Its content is in turn subdivided into three subsections. The first two, which are at first sight uncorrelated, are devoted to the basic properties of certain 4D spaces and to a concise description of the subdivision scheme. The third clarifies the connection between the previous ones discussing structure, properties and a suitable basis of a class of 4D spaces which can be constructed by means of some steps of the proposed subdivision scheme.

2.1. Generalized cubics

In this section we recall some necessary material on Bernstein-like bases of suitable 4D spaces. Given a real function $\boldsymbol{f}$,[2] we denote with $\dot{\boldsymbol{f}}$ the derivative with respect to the (local) variable t and with $\boldsymbol{f}'$ the derivative with respect to the (global)

[2] Throughout the paper, bold will be used to denote functions, in contrast with non-bold which will be used to denote points or sequences of points.