Journal of Computational and Applied Mathematics 233 (2010) 1685–1696

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Running error for the evaluation of rational Bézier surfaces

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ARTICLE INFO

Article history:
Received 12 November 2007
Received in revised form 10 April 2008

MSC:
65G50
65D17
41A20

Keywords:
Roundoff error
Rational surfaces
Running error
Forward error

ABSTRACT

Error analysis of the usual method to evaluate rational Bézier surfaces is performed. The corresponding running error analysis is also carried out and the sharpness of our running error bounds is shown. We also modify the evaluation algorithm to include such error bounds without increasing significantly its computational cost.

1. Introduction

The evaluation of rational Bézier surfaces is an important task in the field of Geometric Modeling (cf. [1] and [2]) as well as in other fields such as finite elements (cf. [3] and [4]). The usual method to evaluate rational Bézier surfaces uses the projection operator Π through the Bernstein basis. In [5] it was proved that this basis presents optimal stability properties and the advantages of this algorithm over other evaluation algorithms of nested type and with lower complexity were shown. However, in some circumstances overflow or underflow problems can appear and an alternative algorithm was proposed in [5].

In this paper we perform a forward error analysis of the usual method to evaluate rational Bézier surfaces. As far as we know, there is no such error analysis in the literature. In fact, the error analysis in the simpler case of tensor product surfaces has been performed very recently (see [6]), and it also includes the running error analysis. Another main contribution of this paper is the running error analysis of the usual method to evaluate rational surfaces, providing *a posteriori* error bounds. We also modify the algorithm to include an estimation of such error bounds at the same time as the evaluation without increasing significantly its computational cost. The error bound obtained with the running error analysis will be more realistic than the “a priori” bounds of the algorithms. We also include illustrative numerical experiments confirming the theoretical results and the accuracy of the error bounds.

In Section 2 we perform the forward error analysis of the algorithm and in Section 3 the running error analysis. Finally, in Section 4 we present some numerical experiments and the conclusions.

Let us now introduce some basic notations. Let

$$F(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} f_{ij}\,\frac{w_{ij}\,b_i^m(x)\,b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}\,b_i^m(x)\,b_j^n(y)},\qquad (x,y)\in[0,1]\times[0,1], \tag{1}$$

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doi:10.1016/j.cam.2009.02.023

be a rational Bézier function with $(f_{ij})_{0\le i\le m}^{0\le j\le n}$ a sequence in $\mathbb{R}$ and $(w_{ij})_{0\le i\le m}^{0\le j\le n}$ a sequence formed by strictly positive weights.

Let us now introduce some standard notations in error analysis. Given $a \in \mathbb{R}$, the computed element in floating point arithmetic will be denoted by either $fl(a)$ or by $\widehat{a}$. As usual, to investigate the effect of rounding errors we use either the model

$$fl(a\ op\ b) = (a\ op\ b)\,(1+\delta), \quad |\delta| \le u, \tag{2}$$

or the model

$$fl(a\ op\ b) = \frac{a\ op\ b}{1+\delta}, \quad |\delta| \le u, \tag{3}$$

with u the unit roundoff and op any of the elementary operations $+, -, \times, /$ (see pages 44–45 of [7] for more details). Given $k \in \mathbb{N}_0$ such that $ku < 1$, let us define

$$\gamma_k := \frac{ku}{1-ku} = ku + \mathcal{O}(u^2). \tag{4}$$

In our error analysis we shall deal with quantities satisfying the condition that their absolute values are bounded above by γ_k. Following [7] we denote by θ_k such quantities and take into account that, by Lemmas 3.3 and 3.1 of [7], the following properties hold:

$$(1+\theta_k)\,(1+\theta_j) = 1+\theta_{k+j}, \tag{5}$$

$$\frac{1+\theta_k}{1+\theta_j} = \begin{cases} 1+\theta_{k+j}, & j \le k, \\ 1+\theta_{k+2j}, & j > k. \end{cases} \tag{6}$$

2. Error analysis of the evaluation algorithm

In CAGD the usual algorithm for evaluating a rational function (1) considers the auxiliary vectorial function

$$\widetilde{F}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{pmatrix} w_{ij} f_{ij} \\ w_{ij} \end{pmatrix} b_i^m(x)\, b_j^n(y).$$

This algorithm can be written explicitly in the following way:

Algorithm 1. Let $F(x,y)$ be the rational function given by (1) and (x,y) be a fixed point in $[0,1]\times[0,1]$. Then, performing

1. For $i = 0 : m$
   For $j = 0 : n$
   $f_{ij}^{00} = w_{ij} f_{ij}, \quad w_{ij}^{00} = w_{ij}$
   End-For
   End-For
2. For $i = 0 : m$
   For $r = 1 : n$
   For $j = 0 : (n-r)$
   $f_{ij}^{0r} = (1-y) f_{ij}^{0,r-1} + y f_{i,j+1}^{0,r-1}, \quad w_{ij}^{0r} = (1-y)\, w_{ij}^{0,r-1} + y\, w_{i,j+1}^{0,r-1}$
   End-For
   End-For
   End-For
3. For $r = 1 : m$
   For $i = 0 : (m-r)$
   $f_{i0}^{rn} = (1-x) f_{i0}^{r-1,n} + x f_{i+1,0}^{r-1,n}, \quad w_{i0}^{rn} = (1-x)\, w_{i0}^{r-1,n} + x\, w_{i+1,0}^{r-1,n}$
   End-For
   End-For
4. $output = \frac{f_{00}^{mn}}{w_{00}^{mn}}$

we have $output = F(x,y)$.

The following result states the forward error analysis of this algorithm.

Theorem 1. *Let us consider a basis*

$$b := \left(\frac{w_{ij}\, b_i^m(x)\, b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}\, b_i^m(x)\, b_j^n(y)} \right)_{0\le i\le m}^{0\le j\le n} \tag{7}$$