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Orthogonal polynomials and measures on the unit circle. The Geronimus transformations

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ABSTRACT

In this paper we analyze a perturbation of a nontrivial positive measure supported on the unit circle. This perturbation is the inverse of the Christoffel transformation and is called the Geronimus transformation. We study the corresponding sequences of monic orthogonal polynomials as well as the connection between the associated Hessenberg matrices. Finally, we show an example of this kind of transformation.

1. Introduction

The study of orthogonal polynomials with respect to a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ was started by Szegő in several papers published from 1915 to 1925 (see [1]). Later on, Geronimus [2] extended this theory to a more general situation.

If υ is a linear functional in the linear space Λ of the Laurent polynomials $\left(\Lambda = \text{span}\,\{z^n\}_{n\in\mathbb{Z}}\right)$ such that υ is Hermitian, i.e. $c_n = \langle \upsilon, z^n\rangle = \overline{\langle \upsilon, z^{-n}\rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$, then a bilinear functional associated with υ can be introduced in the linear space $\mathbb{P}$ of polynomials with complex coefficient as follows:

$$(p(z), q(z))_\upsilon = \left\langle \upsilon, p(z)\bar{q}(z^{-1})\right\rangle \tag{1}$$

where $p, q \in \mathbb{P}$.

The Gram matrix associated with this bilinear functional in terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$ is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{2}$$

a Toeplitz matrix [3].

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The linear functional is said to be quasi-definite if the principal leading submatrices of **T** are non-singular. If such matrices have positive determinant, then the linear functional is said to be positive definite. Every positive definite linear functional has an integral representation

$$\langle \upsilon, p(z)\rangle = \int_{\mathbb{T}} p(z)\mathrm{d}\mu(z), \tag{3}$$

where μ is a nontrivial positive Borel measure supported on the unit circle (see [2–5]).

If υ is a quasi-definite linear functional then a unique sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that

$$(P_n, P_m)_\upsilon = \mathbf{k}_n \delta_{n,m}, \tag{4}$$

can be introduced, where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with υ.

This polynomial sequence satisfies two equivalent recurrence relations due to Szegő (see [2,3,5,1])

$$P_{n+1}(z) = zP_n(z) + P_{n+1}(0)P_n^*(z), \tag{5}$$

$$P_{n+1}(z) = \left(1 - |P_{n+1}(0)|^2\right) zP_n(z) + P_{n+1}(0)P_{n+1}^*(z), \tag{6}$$

the forward and backward recurrences, respectively, where $P_n^*(z) = z^n \bar{P}_n(z^{-1})$ is the so-called reversed polynomial. On the other hand, from (5) and (6) we deduce

$$zP_n(z) = \sum_{j=0}^{n+1} \lambda_{n,j} P_j(z), \tag{7}$$

with

$$\lambda_{n,j} = \begin{cases} 1 & \text{if } j = n+1, \\ \dfrac{\mathbf{k}_n}{\mathbf{k}_j} P_{n+1}(0)\overline{P_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(see [6,5]). Thus, the matrix representation of the linear operator $h : \mathbb{P} \mapsto \mathbb{P}$, the multiplication by z, in terms of the basis $\{P_n\}_{n\geqslant 0}$ is

$$zP(z) = \mathbf{H}_P P(z),$$

where $P(z) = [P_0(z), P_1(z), \ldots, P_n(z), \ldots]^t$ and $\mathbf{H}_P$ is a lower Hessenberg matrix with entries $\lambda_{j,k}$ defined in (8).

Finally, in terms of the moments $\{c_n\}_{n\geqslant 0}$ an analytic function

$$C(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n \tag{9}$$

can be introduced. If υ is a positive definite linear functional, then C is analytic in the open unit disk and $\mathfrak{Re}\,(C(z)) > 0$ therein. In such a case C is said to be a Carathéodory function and it can be represented as a Riesz–Herglotz transform of the positive measure μ introduced in (3) (see [2,4,5]):

$$C(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} \mathrm{d}\mu(w).$$

Following some perturbations of the measure μ we have studied the behavior of the corresponding Carathéodory functions (see [7]) as well as the Hessenberg matrices associated with the corresponding sequence of orthogonal polynomials in three cases:

(i) If $\mathrm{d}\tilde{\mu} = |z-\alpha|^2 \mathrm{d}\mu$, $|z| = 1$, then the so-called canonical Christoffel transformation appears. In [21] and [8] we have studied the connection between the associated Hessenberg matrices using the *QR* factorization. The iteration of the canonical Christoffel transformation has been analyzed in [9–11].

(ii) If $\mathrm{d}\tilde{\mu} = \mathrm{d}\mu + \boldsymbol{m}\delta(z - z_0)$, $|z_0| = 1$, $\boldsymbol{m} \in \mathbb{R}_+$, then the so-called canonical Uvarov transformation appears. In [21] and [7] we have studied the connection between the corresponding sequences of monic orthogonal polynomials as well as the associated Hessenberg matrices using the *LU* and *QR* factorizations. The iteration of the canonical Uvarov transformation has been studied in [2] and [11].

(iii) If $\mathrm{d}\tilde{\mu} = \frac{1}{|z-\alpha|^2}\mathrm{d}\mu$, $|z| = 1$, and $|\alpha| > 1$, then a special case of the Geronimus transform has been analyzed in [12]. In particular, the relation between the corresponding sequences of monic orthogonal polynomials and the associated Hessenberg matrices is stated. A more general framework is presented in [13].

These three examples of canonical spectral transforms are the analogues on the unit circle of the canonical spectral transforms on the real line considered by several authors (see [14–16] and [17]) in connection with bispectral problems and *LU*, *UL*, and *QR* factorizations of Jacobi matrices, i.e. symmetric and tridiagonal matrices with real entries. An extension of the canonical Christoffel transformation for a general inner product is done in [18].