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Cascadic multilevel methods for ill-posed problems

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Dedicated to Bill Gragg on the occasion of his 70th birthday

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ABSTRACT

Multilevel methods are popular for the solution of well-posed problems, such as certain boundary value problems for partial differential equations and Fredholm integral equations of the second kind. However, little is known about the behavior of multilevel methods when applied to the solution of linear ill-posed problems, such as Fredholm integral equations of the first kind, with a right-hand side that is contaminated by error. This paper shows that cascadic multilevel methods with a conjugate gradient-type method as basic iterative scheme are regularization methods. The iterations are terminated by a stopping rule based on the discrepancy principle.

1. Introduction

Bill Gragg's many contributions to scientific computing include work on ill-posed problems [1], iterative solution of symmetric, possibly indefinite linear systems of equations [2], and Toeplitz matrices [3,4]. This paper is concerned with all these topics.

Many problems in science and engineering can be formulated as a Fredholm integral equation of the first kind,

$$\int_{\Omega} \kappa(t,s)x(s)\mathrm{d}s = b(t), \quad t \in \Omega. \tag{1}$$

Here Ω denotes a compact Jordan measurable subset of $\mathbb{R} \times \cdots \times \mathbb{R}$, and the kernel κ and right-hand side b are smooth functions on $\Omega \times \Omega$ and Ω, respectively. The computation of the solution x of (1) is an ill-posed problem because (i) the integral equation might not have a solution, (ii) the solution might not be unique, and (iii) the solution – if it exists and is unique – does not depend continuously on the right-hand side. The computation of a meaningful approximate solution of (1) in finite precision arithmetic therefore is delicate; see, e.g., [5] or [6] for discussions on the solution of ill-posed problems. In the present paper, we assume that (1) is consistent and has a solution in a Hilbert space $\mathcal{X}$ with norm $\|\cdot\|$. For instance, $\mathcal{X}$ may be $L_2(\Omega)$. Often one is interested in determining the unique solution of minimal norm. We denote this solution by $\hat{x}$.

In applications, generally, not b, but a corrupted version, which we denote by b^δ, is available. We assume that a constant $\delta > 0$ is known, such that the inequality

$$\|b^\delta - b\| \le \delta \tag{2}$$

holds. The difference $b^\delta - b$ may, for instance, stem from measurement errors and is referred to as "noise".

Our task is to determine an approximate solution x^δ of

$$\int_{\Omega} \kappa(t,s)x(s)\mathrm{d}s = b^\delta(t), \quad t \in \Omega, \tag{3}$$

such that x^δ provides an accurate approximation of $\hat{x}$. Eq. (3) is not required to be consistent.

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In operator notation, we express (1) and (3) as

$$Ax = b \tag{4}$$

and

$$Ax = b^\delta, \tag{5}$$

respectively. The operator $A : \mathcal{X} \to \mathcal{Y}$ is compact, where $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces. Thus, A has an unbounded inverse and may be singular. The right-hand side b is assumed to be in the range of A, denoted by $\mathcal{R}(A)$, but b^δ generally is not.

We seek to determine an approximation of the minimal-norm solution $\hat{x}$ of (4) by first replacing the operator A in (5) by an operator A_{reg} that approximates A and has a bounded inverse on $\mathcal{Y}$, and then solving the modified equation so obtained,

$$A_{\text{reg}}x = b^\delta. \tag{6}$$

The replacement of A by A_{reg} is referred to as regularization and A_{reg} as a regularized operator. We would like to choose A_{reg} so that the solution x^δ of (6) is a meaningful approximation of $\hat{x}$.

One of the most popular regularization methods is Tikhonov regularization, which in its simplest form is defined by

$$(A^*A + \lambda I)x = A^*b^\delta, \tag{7}$$

i.e., $(A_{\text{reg}})^{-1} = (A^*A + \lambda I)^{-1}A^*$. Here I is the identity operator, $\lambda > 0$ is a regularization parameter, and A^* denotes the adjoint of A. The latter determines how sensitive the solution x^δ of (7) is to perturbations in the right-hand side b^δ and how close x^δ is to the solution $\hat{x}$ of (4); see, e.g., [5,6] for discussions on Tikhonov regularization.

For any fixed $\lambda > 0$, Eq. (7) is a Fredholm integral equation of the second kind and, therefore, the computation of its solution is a well-posed problem. Several two-level and multilevel methods for the solution of the Tikhonov equation (7) have been described in the literature; see, e.g., [7–11]. For a large number of ill-posed problems, these methods determine accurate approximations of the solution of the Tikhonov equation (7) faster than standard (one-level) iterative methods.

The cascadic multilevel method of the present paper is applied to the unregularized problem (5). Regularization is achieved by restricting the number of iterations on each level using the discrepancy principle, defined in Section 2. Thus, the operator A_{reg} associated with the cascadic multilevel method is defined implicitly. For instance, let the basic iterative scheme be CGNR (the conjugate gradient method applied to the normal equations). We apply CGNR on the coarsest discretization level until the computed approximate solution satisfies the discrepancy principle. Then the coarsest-level solution is prolongated to the next finer discretization level and iterations with CGNR are carried out on this level until the computed approximate solution satisfies the discrepancy principle. The computations are continued in this manner until an approximate solution on the finest discretization level has been found that satisfies the discrepancy principle. We remark that if the iterations are not terminated sufficiently early, then the error in b^δ may propagate to the computed approximate solution and render the latter a useless approximation of $\hat{x}$. We establish in Section 3 that the CGNR-based cascadic multilevel method is a regularization method in a well-defined sense.

The application of CGNR as basic iterative method in the multilevel method is appropriate when A is not self-adjoint. The computed iterates live in $\mathcal{R}(A^*)$ and therefore are orthogonal to $\mathcal{N}(A)$, the null space of A.

When A is self-adjoint, the computational work often can be reduced by using an iterative method of conjugate gradient (CG) type different from CGNR as basic iterative method. Section 3 also describes multilevel methods for self-adjoint ill-posed problems based on a suitable minimal residual method.

The application of multigrid methods directly to the unregularized problem (5) recently also has been proposed by Donatello and Serra-Capizzano [12], who with computed examples show the promise of this approach. The regularization properties of the multigrid methods used are not analyzed in [12].

Cascadic multilevel methods typically are able to determine an approximate solution of (5) that satisfies the discrepancy principle with less arithmetic work than application of the CG-type method, which is used for the basic iterations, on the finest level only. We refer to the latter method as a one-level CG-type method, or simply as a CG-type method. A cascadic Landweber-based iterative method for nonlinear ill-posed problems has been analyzed by Scherzer [13]. Numerical examples reported in [13] show this method to require many iterations.

Multilevel methods have for many years been applied successfully to the solution of well-posed boundary value problems for partial differential equations; see, e.g., [14] and the references therein. In particular, a CG-based cascadic multigrid method has been analyzed in [15]. However, the design of multilevel methods for this kind of problem differs significantly from multilevel methods for ill-posed problems. This depends on that highly oscillatory eigenfunctions, which need to be damped, in the former problems are associated with eigenvalues of large magnitude, while they are associated with eigenvalues of small magnitude for the latter problems.

This paper is organized as follows. Section 2 reviews CG-type methods and the discrepancy principle. In particular, we discuss the regularization properties of CG-type methods. Cascadic multilevel methods based on different CG-type methods are described in Section 3, where also regularization properties of these methods are shown. Section 4 presents a few computed examples and concluding remarks can be found in Section 5.