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Qualitative behavior and exact travelling wave solutions of the Zhiber–Shabat equation

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ABSTRACT

In this paper, the qualitative behavior and exact travelling wave solutions of the Zhiber–Shabat equation are studied by using qualitative theory of polynomial differential system. The phase portraits of system are given under different parametric conditions. Some exact travelling wave solutions of the Zhiber–Shabat equation are obtained. The results presented in this paper improve the previous results.

1. Introduction

Recently, Wazwaz [1] investigated the nonlinear Zhiber–Shabat equation

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0, \tag{1.1}$$

where p, q and r are arbitrary constant. For $q = r = 0$, Eq. (1.1) reduces to the Liouville equation. For $r = 0$, Eq. (1.1) reduces to the sinh-Gordon equation. However, for $q = 0$, Eq. (1.1) gives the well-known Dodd–Bullough–Mikhailov equation. Moreover, for $p = 0, q = -1, r = 1$, we obtain the Tzitzeica–Dodd–Bullough equation. The aforementioned equations play a significant role in many scientific applications.

In [1], some travelling wave solutions were established by using the tanh method and extended tanh method. However, it is very important to understand the dynamical behavior of solutions for travelling wave equation. In the past years, many authors have made tremendous efforts to investigate the dynamical behavior of solutions for travelling wave models, and many interesting research results have been obtained [2–5].

In [6], the bifurcation behaviors of travelling wave solutions of the Zhiber–Shabat equation are studied. Moreover, the authors obtain the conditions under which smooth and non-smooth travelling wave solutions exist. But the authors did not study the qualitative behavior of the travelling wave solutions in the case of degenerate singular points.

In this paper, the qualitative behavior and exact travelling wave solutions of the Zhiber–Shabat equation are studied by using qualitative theory of polynomial differential system. The phase portraits of system are given under different parametric conditions. Some exact travelling wave solutions of the Zhiber–Shabat equation are obtained. Our results improve the previous results of [1] and [6].

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2. Some preliminaries

To study the qualitative behavior of the Zhiber–Shabat equation, we first need to introduce some notations and propositions.

We denote by $P_n(\mathbf{R}^2)$ the set of polynomial vector fields on $\mathbf{R}^2$ of the form $X(x, y) = (P(x, y), Q(x, y))$ where P and Q are real polynomials in the variables x and y of degree at most n (with $n \in \mathbf{N}$). As usual $\mathbf{N}$ denotes the set of positive integers.

A point $p \in \mathbf{R}^2$ is said to be a *singular point* of the vector field $X = (P, Q)$ if $P(p) = Q(p) = 0$. We recall first some results which hold when P and Q are analytic functions in a neighborhood of p. As usual P_x denotes the partial derivative of P with respect to the variable x.

If $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p)$ and $T = P_x(p) + Q_y(p)$, then the singular point p is said to *non-degenerate* if $\Delta \neq 0$. Then p is an isolated singular point. Moreover, p is a *saddle* if $\Delta < 0$, a *node* if $T^2 > 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < \Delta$.

The singular point p is called *hyperbolic* if the two eigenvalues of the Jacobian matrix $DX(p)$ have nonzero real part. So, the hyperbolic points are non-degenerate ones except the weak focus and the centers.

A degenerate singular point p (i.e. $\Delta = 0$) with $T \neq 0$ is called *semi-hyperbolic*, and p is isolated in the set of all singular points. Now we summarize the results on semi-hyperbolic singular points that we shall need in this paper; for a proof see Theorem 65 of [7].

Proposition 1. *Let $(0, 0)$ be an isolated point of the vector field $(F(x, y), y + G(x, y))$, where F and G are analytic functions in a neighborhood of the origin at least with quadratic terms in the variables x and y. Let $y = g(x)$ is the solution of the equation $y + G(x, y) = 0$ in a neighborhood of $(0, 0)$. Assume that the development of the function $f(x) = F(x, g(x))$ is of the form $f(x) = \mu x^m + HOT$ (Higher Order Terms), where $m \geq 2$ and $\mu \neq 0$. When m is odd, then (0.0) is either an unstable node, or a saddle depending if $\mu > 0$, or $\mu < 0$, respectively. In the case of the saddle the stable separatrices are tangent to the x-axis. If m is even, then $(0, 0)$ is a saddle-node, i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector. The stable separatrix is tangent to the positive (respectively negative) x-axis at $(0, 0)$ according to $\mu < 0$ (respectively $\mu > 0$). Then two unstable separatrices are tangent to the y-axis at $(0, 0)$.*

The singular points which are non-degenerate or semi-hyperbolic are called *elementary*. When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is *nilpotent*. Now we summarize the results on nilpotent singular points that we shall need. For a proof see [8], or Theorem 66 and 67 and the simplified scheme of Section 22.3 of [7].

Proposition 2. *Let $(0, 0)$ be an isolated point of the vector field $(y + F(x, y), G(x, y))$, where F and G are analytic functions in a neighborhood of the origin at least with quadratic terms in the variables x and y. Let $y = f(x)$ is the solution of the equation $y + F(x, y) = 0$ in a neighborhood of $(0, 0)$. Assume that the development of the function $G(x, f(x))$ is of the form $Kx^k + HOT$ (Higher Order Terms) and $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = Lx^\lambda + HOT$ with $K \neq 0$, $k \geq 2$ and $\lambda \geq 1$. Then the following statements hold.*

(1) *If k is even and*
(1.a) *$k > 2\lambda + 1$, then the origin is a saddle-node. Moreover the saddle-node has one separatrix tangent to the semi-axis $x < 0$, and other two separatrices tangent to the semi-axis $x > 0$.*
(1.b) *$k < 2\lambda + 1$ or $\Phi \equiv 0$, then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the positive x-axis.*
(2) *If k is odd and $K > 0$, then the origin is a saddle. Moreover, the saddle has two separatrices tangent to the semi-axis $x < 0$, and other two separatrices tangent to the semi-axis $x > 0$.*
(3) *If k is odd, $K < 0$ and*
(3.a) *λ even, $k = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ even and $k > 2\lambda + 1$, then the origin is a stable (unstable) node if $L < 0 (L > 0)$, having all the orbits tangent to x-axis at $(0, 0)$.*
(3.b) *λ odd, $k = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ odd and $k > 2\lambda + 1$, then the origin is an elliptic-saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector. Moreover, one separatrix of the elliptic-saddle is tangent to the semi-axis $x < 0$, and the other to the semi-axis $x > 0$.*
(3.c) *$k = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) < 0$, or $k < 2\lambda + 1$, then the origin is a focus or a center, and if $\Phi(x) \equiv 0$ then the origin is a center.*

Finally, if the Jacobian matrix at p is identically zero and p is isolated inside the set of all singular points, then we say that p is *linearly zero*. The study of its local phase portraits needs a special treatment (directional blow-ups), see for more details [7,9]. In order to state our results, the following proposition is needed.

Proposition 3. *Let $(0, 0)$ be an isolated point of the vector field $(X_n(x, y) + \Phi(x, y), Y_n(x, y) + \Psi(x, y))$, where X_n and Y_n are homogeneous polynomials in the variables x and y of degree n (with $n \in \mathbf{N}$), $\Phi(x, y)$ and $\Psi(x, y)$ are analytic functions in a neighborhood of the origin with terms of degree higher than n in the variables x and y. Let $y = ux$, if $X_n(1, u_k) = x^{-n}X_n(x, u_kx) \neq 0$, and the function*

$$G(\theta) = \cos\theta Y_n(\cos\theta, \sin\theta) - \sin\theta X_n(\cos\theta, \sin\theta) \tag{2.1}$$