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A new difference scheme with high accuracy and absolute stability for solving convection–diffusion equations[☆]

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ABSTRACT

In this paper, we use a semi-discrete and a padé approximation method to propose a new difference scheme for solving convection–diffusion problems. The truncation error of the difference scheme is $O(h^4 + \tau^5)$. It is shown through analysis that the scheme is unconditionally stable. Numerical experiments are conducted to test its high accuracy and to compare it with Crank–Nicolson method.

1. Introduction

Consider the convection–diffusion equation

$$\frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \tag{1.1}$$

subject to the initial condition

$$u(x, 0) = g(x), \quad 0 \le x \le 1$$

and boundary conditions

$$u(0, t) = 0, \quad t > 0.$$
$$u(1, t) = 0, \quad t > 0,$$

where the parameter γ is the viscosity coefficient and ε is the phase speed, and both are assumed to be positive. g is a given function of sufficient smoothness. This equation may be seen in computational hydraulics and fluid dynamics modeling convection–diffusion of quantities such as mass, heat, energy, vorticity, etc [1].

There has been much work on computing a finite difference approximation solution of equation (1.1), see [2–4]. We focus our attention on a method based on the high-order compact (HOC) finite difference discretization of equation (1.1)

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only with respect to the space variable. This type of discretization yields a system of ordinary differential equation. The solution of this system requires the computation of $e^{\tau A^{-1}B}\mathbf{y}$ for some vector $\mathbf{y}$, where τ is the time step-size, A and B are large Toeplitz matrixes. There are various methods to compute an approximation of $e^{\tau A^{-1}B}\mathbf{y}$. In [5–7,12], some approaches based on the Krylov subspace method were proposed. The restrictive Taylor's approximation method has been presented in [8,12]. In most of the cases, the accuracy of the difference schemes constructed by using the above methods is second order in time direction and second or fourth order in space direction. In this paper,we use padé approximation method to give an expression to compute the value of $e^{\tau A^{-1}B}$. So we get a new difference scheme for solving convection–diffusion equation (1.1) and the truncation error is $O(\tau^5 + h^4)$. Then the numerical results of our difference scheme for computing the approximate solution of Eq. (1.1) at some given time levels are compared with that of Crank–Nicolson scheme.

The present paper is organized as follows. In Section 2, we define the difference scheme and discuss the accuracy. In Section 3, we do stability analysis. Some numerical examples are presented in Section 4 and concluding remarks are given in Section 5.

2. Proposition of the difference scheme

We subdivide the interval $0 \le x \le 1$ into n equal subintervals by the grid points $x_i = ih$, $i = 0(1)n$, where $h = 1/n$,(n is a positive integer). The mesh function $u(ih, k\tau)$ is written as u_i^k at grid point $(ih, k\tau)$.

We start by examining the one-dimensional steady convection equation

$$-\gamma\frac{d^2u}{dx^2} + \varepsilon\frac{du}{dx} = f, \tag{2.1}$$

where f is a function of x. Using the techniques outlined in [9,10], it is easy to derive a three-point fourth-order compact scheme for Eq. (2.1) as

$$-\left(\gamma + \frac{\varepsilon^2h^2}{12\gamma}\right)\delta_x^2 u_i + \varepsilon\delta_x u_i = \left(1 + \frac{h^2}{12}(\delta_x^2 - \frac{\varepsilon}{\gamma}\delta_x)\right)f_i + O(h^4), \tag{2.2}$$

where δ_x^2 and δ_x are the second-order and first-order center difference operators.

For convenience, we define two difference operators as follows

$$L_x = 1 + \frac{h^2}{12}(\delta_x^2 - \frac{\varepsilon}{\gamma}\,\delta_x), \qquad A_x = -\left(\gamma + \frac{\varepsilon^2h^2}{12\gamma}\right)\delta_x^2 + \varepsilon\delta_x.$$

Eq. (2.2) can then be formulated symbolically as

$$L_x^{-1}A_x u_i = f_i + O(h^4). \tag{2.3}$$

A fourth-order semi-discrete approximation to the unsteady convection–diffusion equation in (1.1) can be obtained by replacing f with $-\frac{\partial u}{\partial t}$ in (2.3)

$$L_x^{-1}A_x u_i^k = -\frac{\partial u_i^k}{\partial t} + O(h^4). \tag{2.4}$$

Let

$$v_i^k = \frac{\partial u_i^k}{\partial t}. \tag{2.5}$$

Then we have

$$L_x^{-1}A_x u_i^k = -v_i^k + O(h^4). \tag{2.6}$$

Neglecting the high-order term $O(h^4)$ of (2.6) and then rewriting it as follows

$$\begin{aligned}&\left(\frac{1}{12} + \frac{h\varepsilon}{24\gamma}\right)v_{i-1}^k + \frac{5}{6}v_i^k + \left(\frac{1}{12} - \frac{h\varepsilon}{24\gamma}\right)v_{i+1}^k \\ &= \left(\frac{\gamma}{h^2} + \frac{\varepsilon^2}{12\gamma} + \frac{\varepsilon}{2h}\right)u_{i-1}^k + \left(-\frac{2\gamma}{h^2} - \frac{\varepsilon^2}{6\gamma}\right)u_i^k + \left(\frac{\gamma}{h^2} + \frac{\varepsilon^2}{12\gamma} - \frac{\varepsilon}{2h}\right)u_{i+1}^k.\end{aligned} \tag{2.7}$$

Along time level t, we denote $w(x_i, t)$ by $w_i(t)$, where w is u or v.

In matrix notation, (2.7) can be written as:

$$\begin{cases}\mathbf{AV}(t) = \mathbf{BU}(t),\\ \mathbf{U}(0) = \mathbf{U}_0.\end{cases} \tag{2.8}$$