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A rational spectral method for the KdV equation on the half line☆

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ABSTRACT

A Petrov–Galerkin method using orthogonal rational functions is proposed for the Korteweg–de Vries (KdV) equation on the half line with initial-boundary values. The nonlinear term and the right-hand side term are treated by Chebyshev rational interpolation explicitly, and the linear terms are computed with the Galerkin method implicitly. Such an approach is applicable using fast algorithms. Numerical results are presented for problems with both exponentially and algebraically decaying solutions, respectively, highlighting the performance of the proposed method.

1. Introduction

Spectral methods for differential equations on unbounded domains have been concentrated on recently. For problems on $\Lambda = (0, +\infty)$, there are three basic ways: one is to expand in the Laguerre functions [1], another is to map the semi-infinite interval into a finite one and then expand in orthogonal polynomials on the mapped interval (see eg. [2]), and a third way is to use algebraically mapped Legendre or Chebyshev functions so-called a Legendre or Chebyshev rational method (see eg. [3–7]).

Compared with the the first two methods, we prefer the third way. It is attractive since it is easy to implement and close to Legendre and Chebyshev spectral methods. Also, one can provide a relatively simple analysis, while the second way often results in complicated mapped equations and cumbersome analysis. It is believed that the last two ways are equivalent. However, the reader may find the differences between them after finishing the paper. When applied to nonlinear equations, a major handicap is the lack of efficient fast transform for Laguerre approximations. For these reasons, the rational methods are adopted for different problems, such as using Chebyshev rational functions in the method of matched asymptotic expansions [8]; using Legendre rational functions on the whole line for a Dirac equation [9]; and utilizing a Chebyshev rational spectral method for the Helmholtz equation in a sector on the surface of a sphere to defeat corner singularities [10], etc.

The close relationship between rational polynomials and algebraic polynomials works for introducing the so-called Chebyshev–Legendre method [11,12] for the rational approximation. The combined method on finite intervals has received a great success in solving partial differential equations [13–15]. It is in essence a collocation method in framework of Galerkin methods. The interpolant using the Gauss–Chebyshev nodes is represented as a finite series of Legendre polynomials, and then can be implemented in the framework of Galerkin methods. Since it is a collocation method, the computation cost for nonlinearity is cut down and further reduced by a fast Legendre transform (FLT, [16]).

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Li [17] first combined the Chebyshev–Legendre method with the rational spectral method to solve the Burgers equation on the half line. Therein the nonlinear term is treated with the Chebyshev–Gauss rational interpolation explicitly with the help of FLT, and the linear terms are computed with Galerkin method implicitly. This method is followed in [18] for the Benjamin–Bona–Mahony equation on the half line. Both show such an approach is of economization and of numerical stability. Hence it may be of interest to extend the method to the following KdV equation with initial-boundary values on the half line:

$$\begin{cases} \partial_t U(x,t) + U(x,t)\partial_x U(x,t) + \partial_x^3 U(x,t) = f(x,t), & x \in \Lambda, t \in (0,T], \\ U(0,t) = \lim\limits_{x\to+\infty} U(x,t) = \lim\limits_{x\to+\infty} \partial_x U(x,t) = 0, & t \in [0,T], \\ U(x,0) = U_0(x), & x \in \Lambda. \end{cases} \tag{1.1}$$

The well-posedness of (1.1) can be established in vanishing viscosity methods [19]. Guo and Shen [5] proposed a Legendre rational spectral method in Galerkin form for the problem (1.1), where the solution for the numerical scheme processes certain conservation properties which are satisfied by the solution of the Korteweg–de Vries equation with $f \equiv 0$. Their semi-discrete scheme admitted an $O(N^{3-r})$ rate of the convergence in L^2-norm. But there is no temporal discretization for it, which is also a motivation of this work.

In this work, we propose a combined Petrov–Galerkin scheme using orthogonal rational polynomials for the problem (1.1). We follow [20,13] using the Petrov–Galerkin method to overcome the antisymmetry of the third-order differentiation operator. See Shen [21], Zhao et al. [22] and Yuan et al. [23] for more on the Petrov–Galerkin for third- and fifth-order differential equations on finite intervals. So the proposed scheme here is quite different from that in [17] with a direct Chebyshev rational interpolation treatment for the nonlinear term and the right-hand side term replaced by a modified one. This is also motivated by stability and convergence.

The outline of the rest of this paper is as follows. After Introduction, some notations are given in the next section. There are schemes and algorithm descriptions in Section 2. In Section 3, we also present the some basic properties of the Legendre rational functions. Some projections and Chebyshev–Gauss rational interpolation operator with their properties are given in the same section. They play an important role in the error analysis. Section 4 is devoted to illustrate how the Petrov–Galerkin method works via a linear problem of third-order. In Section 5 error estimates for the semi-discrete and the fully-discrete scheme are given. Some numerical results are reported in Section 6. And there are some concluding remarks in Section 7.

2. Schemes and algorithm descriptions

This section is devoted to rational spectral schemes for the KdV equation (1.1) on the half line. Denote $\Lambda = (0, +\infty)$. Let $L_k(x)$ be the Legendre polynomial of order k. We define the Legendre rational functions [5] of degree k by

$$R_k(x) = \frac{\sqrt{2}}{x+1} L_k\left(\frac{x-1}{x+1}\right), \quad k = 0, 1, \ldots.$$

According to the orthogonality of Legendre polynomials,

$$\int_\Lambda R_k(x) R_m(x) \mathrm{d}x = \left(k + \frac{1}{2}\right)^{-1} \delta_{k,m},$$

where $\delta_{k,m}$ is the Kronecker function. Let N be any non-negative integer and

$$\mathrm{R}_N = \mathrm{span}\{R_k(x), k = 0, 1, \ldots, N\}, \qquad \mathrm{R}_N^0 = \mathrm{R}_N \cap H_0^1(\Lambda).$$

A weak form for the problem (1.1) is to find $u \in H^2(\Lambda) \cap H_0^1(\Lambda)$ such that for any $v \in H_0^1(\Lambda)$

$$\begin{cases} (\partial_t u, v) - \frac{1}{2}(u^2, \partial_x v) - (\partial_x^2 u, \partial_x v) = (f, v), & t \in (0,T], \\ (u(0), v) = (U_0, v). \end{cases} \tag{2.1}$$

To get a Petrov–Galerkin method for the KdV equation (1.1) in spectral methods using rational functions, we assume that

$$\lim_{x\to+\infty} (1+x) U(x,t) = 0, \qquad \lim_{x\to+\infty} (1+x)\partial_x U(x,t) = 0, \quad \text{and} \quad \lim_{x\to+\infty} (1+x) f(x,t) \text{ exists.}$$

A semi-discrete spectral scheme for the problem (1.1) is to find $u_N \in \mathscr{R}_N^0$, where $\mathscr{R}_N^0 = \{\omega^{1,0}\phi_n | \phi_n \in \mathbf{R}_{N-1}^0\}$ s.t., for any $v \in \mathbf{R}_{N-1}^0$,

$$\begin{cases} (\partial_t u_N, v) - \frac{1}{2}(\Pi_N^C u_N^2, \partial_x v) - (\partial_x^2 u_N, \partial_x v) = (\Pi_N^C f, v), & t \in (0,T], \\ (u_N(0), v) = (\Pi_N^C U_0, v), \end{cases} \tag{2.2}$$

where $\Pi_N^C u = \omega^{1,0} P_N^C(\omega^{-1,0} u)$, with P_N^C defined in (3.9), $\omega^{1,0} = 2(1+x)^{-1}$, and $\omega^{-1,0} = \frac{1}{2}(1+x)$.

Let τ be the mesh size in t and set $t_k = k\tau (k = 0, 1, \ldots, n_\tau; T = n_\tau \tau)$. For simplicity, we denote $u^k(x) := u(x, t_k)$ by u^k and

$$\hat{u}^k = \frac{u^{k+1} + u^{k-1}}{2}, \qquad u_{\hat{t}}^k = \frac{1}{2\tau}(u^{k+1} - u^{k-1}).$$