Journal of Computational and Applied Mathematics 230 (2009) 762–769

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Anti-periodic solutions for a class of nonlinear nth-order differential equations with delays[☆]

Qiyi Fan [a], Wentao Wang [b], Xuejun Yi [c,*]

[a] Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, PR China

[b] College of Mathematics and Information Engineering, Jiaxing University, Jiaxing, Zhejiang 314001, PR China

[c] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan, 410082, PR China

ARTICLE INFO

Article history:
Received 14 August 2008
Received in revised form 10 January 2009

Keywords:
nth-order differential equations
Delays
Anti-periodic solution
Existence and uniqueness
Leray–Schauder degree

ABSTRACT

In this paper, we use the Leray–Schauder degree theory to establish new results on the existence and uniqueness of anti-periodic solutions for a class of nonlinear nth-order differential equations with delays of the form

$$x^{(n)}(t) + f(t, x^{(n-1)}(t)) + g(t, x(t - \tau(t))) = e(t).$$

1. Introduction

Consider the nonlinear nth-order differential equations with delays of the form

$$x^{(n)}(t) + f(t, x^{(n-1)}(t)) + g(t, x(t - \tau(t))) = e(t), \tag{1.1}$$

where $\tau, e : R \to R$ and $f, g : R \times R \to R$ are continuous functions, τ and e are T-periodic, f and g are T-periodic in its first argument, $n \geq 2$ is an integer, and $T > 0$ is a constant.

Clearly, when $n = 2$ and $f(t, x) = f(x)$, Eq. (1.1) reduces to

$$x'' + f(x'(t)) + g(t, x(t - \tau(t))) = e(t), \tag{1.2}$$

which has been known as the delayed Rayleigh equation. Therefore, we can consider Eq. (1.1) as a high-order delayed Rayleigh equation. Arising from problems in applied sciences, the existence of periodic solutions of Eq. (1.1) has been extensively studied over the past twenty years. We refer the reader to [1–17] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of anti-periodic solutions of Eq. (1.1). Moreover, it is well known that the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [18–24]). Thus, it is worth continuing the investigation of the existence and uniqueness of anti-periodic solutions of Eq. (1.1).

A primary purpose of this paper is to study the existence and uniqueness of anti-periodic solutions of Eq. (1.1). We will establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions of Eq. (1.1). Our results are

☆ This work was supported by the National Natural Science Foundation of China (10771055) and Doctoral Special Fund of Ministry of Education (20060532002).

∗ Corresponding author.
E-mail address: yixuejunhd@yahoo.cn (X. Yi).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.01.005

different from those of the references listed above. In particular, an example is also given to illustrate the effectiveness of our results.

It is convenient to introduce the following assumptions.

(A_0) Assume that there exists a nonnegative constant C_1 such that

$$|f(t,x_1)-f(t,x_2)| \le C_1|x_1-x_2|, \quad \text{for all } t,x_1,x_2 \in R.$$

($\widetilde{A_0}$) Assume that there exists a nonnegative constant C_2 such that

$$f(t,u)=f(u), \qquad C_2|x_1-x_2|^2 \le (x_1-x_2)(f(x_1)-f(x_2)) \quad \text{for all } x_1,x_2,u \in R.$$

(A_1) For all $t,x \in R$,

$$f\left(t+\frac{T}{2},\ -x\right)=-f(t,\ x), \qquad g\left(t+\frac{T}{2},\ -x\right)=-g(t,x), \qquad e\left(t+\frac{T}{2}\right)=-e(t), \qquad \tau\left(t+\frac{T}{2}\right)=\tau(t).$$

2. Preliminary results

For convenience, we introduce a continuation theorem [25] as follows.

Lemma 2.1. *Let Ω be open bounded in a linear normal space X. Suppose that $\widetilde{f}$ is a complete continuous field on $\overline{\Omega}$. Moreover, assume that the Leray–Schauder degree*

$$\deg\{\widetilde{f},\Omega,\ p\} \ne 0, \quad \text{for } p \in X \setminus \widetilde{f}(\partial\Omega).$$

Then equation $\widetilde{f}(x)=p$ has at least one solution in Ω.

Let $u(t): R \longrightarrow R$ be continuous in t. $u(t)$ is said to be anti-periodic on R if,

$$u(t+T)=u(t), \qquad u\left(t+\frac{T}{2}\right)=-u(t), \quad \text{for all } t \in R.$$

We will adopt the following notations:

$$C_T^k := \{x \in C^k(R,\ R), x \quad \text{is } T\text{-periodic}\}, \quad k \in \{0,1,2,\ldots\}$$

$$|x|_q=\left(\int_0^T |x(t)|^q \mathrm{d}t\right)^{1/q}, \qquad |x|_\infty=\max_{t\in[0,T]}|x(t)|, \qquad |x^{(k)}|_\infty=\max_{t\in[0,T]}|x^{(k)}(t)|,$$

$$C_T^{k,\frac{1}{2}} := \left\{x \in C_T^k, x\left(t+\frac{T}{2}\right)=-x(t) \text{ for all } t \in R\right\},$$

which is a linear normal space endowed with the norm $\|.\|$ defined by

$$\|x\|=\max\{|x|_\infty,|x'|_\infty,\ldots,|x^{(k)}|_\infty\}, \quad \text{for all } x \in C_T^{k,\frac{1}{2}}.$$

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2 (*Wirtinger Inequality, See [17]*). *If $x \in C^2(R,R),\ x(t+T)=x(t)$, then*

$$|x'(t)|_2 \le \frac{T}{2\pi}|x''(t)|_2. \tag{2.1}$$

Lemma 2.3. *Assume that one of the following conditions is satisfied:*

(A_2) *Suppose that* (A_0) *holds, and there exists a nonnegative constant b such that*

$$C_1\frac{T}{2\pi}+\frac{b}{2}\frac{T^n}{(2\pi)^{n-1}}<1, \quad \text{and} \quad |g(t,x_1)-g(t,x_2)| \le b|x_1-x_2|, \quad \text{for all } t,x_1,x_2 \in R;$$

(A_3) *Suppose that* ($\widetilde{A_0}$) *hold, and there exists a constant b such that*

$$0 \le b < \frac{2C_2(2\pi)^{n-2}}{T^{n-1}}, \quad \text{and} \quad |g(t,x_1)-g(t,x_2)| \le b|x_1-x_2|, \quad \text{for all } t,x_1,x_2 \in R.$$

Then Eq. (1.1) has at most one anti-periodic solution.