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Global solutions to a class of multi-species reaction-diffusion systems with cross-diffusions arising in population dynamics

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ABSTRACT

In this paper, an n-species strongly coupled cooperating diffusive system is considered in a bounded smooth domain, subject to homogeneous Neumann boundary conditions. Employing the method of energy estimates, we obtain some conditions on the diffusion matrix and inter-specific cooperatives to ensure the global existence and uniform boundedness of a nonnegative solution. The globally asymptotical stability of the constant positive steady state is also discussed. As a consequence, all the results hold true for multi-species Lotka–Volterra type competition model and prey–predator model.

1. Introduction

In 1979, Shigesada, Kawasaki and Teramoto (SKT) proposed the following two-species competing model with Lotka–Volterra type reaction terms [1]:

$$\begin{cases} u_t - \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] = (a_1 - b_1u - c_1v)u & \text{in } \Omega \times (0, \infty), \\ v_t - \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] = (a_2 - b_2u - c_2v)v & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded interval of $\mathbb{R}$, ν is the unit outward normal to $\partial\Omega$, d_i, α_{ij}, a_i, b_i, c_i $(i, j = 1, 2)$ are all positive constants, u_0 and v_0 are nonnegative functions which are not identically zero. In (1.1), u and v are the population densities of two competing species. a_i denotes the intrinsic growth rate of the ith species, b_1 and c_2 account for intra-specific competitions, while c_1 and b_2 are the coefficients for inter-specific competitions. d_i is the diffusion rate of the ith species, α_{ii} is referred as self-diffusion pressure, and α_{ij} $(i \neq j)$ is cross-diffusion pressure of the ith species due to the presence of the jth species.

Diffusion is population pressure due to the mutual interference between the individuals, describing the migration of species to avoid crowds. The term self-diffusion implies the movement of individuals from a higher to lower concentration region. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. The value of the cross-diffusion coefficient may be positive, negative or zero. The term positive cross-diffusion coefficient denotes the movement of the species in the direction of lower concentration of another species and negative cross-diffusion coefficient denotes that one species tends to diffuse in the direction of higher concentration of another species. For more details on the biological backgrounds of this model see also [2,3].

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Problem (1.1) is the two-species SKT competing model which describes spatial segregation of interacting population species in one-dimensional space. The two-species SKT competing system and its overall behavior continue to be of great interest in the literature to both mathematical analysis and real-life modelling, but recently more and more attention has been focused on three or multi-species systems and the SKT model in any space dimension due to their more complicated patterns. Meanwhile, the SKT models with other types of reaction terms are also proposed and investigated [4–11]. The obtained results mainly concentrate on the stability analysis of positive constant steady state and the existence of non-constant positive steady states (stationary patterns) [6–14]. They confirmed the role of cross-diffusion in helping to create patterns. As to the time-dependent solution to SKT model, the local existence was established by H. Amann in a series of important papers [15–17]. The global existence of the weak solution to the strongly coupled two-species competing model was established in [18–20], employing the method of energy estimates, semi-discretization in time and positivity-preserving backward Euler–Galerkin approximation, respectively. Furthermore, [18,19] discussed the case in one-dimensional space while [20] in multi-dimensional space. Papers [21,22] show the global existence of the classical solution of the same model in low-dimensional space by using bootstrap technique. When the dimension of the domain is arbitrary, the same result of global existence was established in [23].

From above we can learn that most of the results about the SKT model is for the two-species competing model. As to the global existence and long time behavior of time-dependent solution to multi-species SKT model or SKT model with generalized reaction terms, very few works are known.

Mathematically, multi-species Lotka–Volterra systems have received a lot of attention. But up to now, the corresponding researches chiefly concern with Lotka–Volterra ODEs and its qualitative analysis such as persistence, permanence and attractability [24–26]. To the best of our knowledge, there are very few results for the multi-species Lotka–Volterra type cross-diffusion models. In view of this, the present paper considers the following n-species Lotka–Volterra system, which is a generalized SKT model:

$$\begin{cases} \partial_t u_i - \Delta\left[\left(d_i + \sum_{j=1}^{n} \alpha_{ij} u_j\right) u_i\right] = \left(a_i - \sum_{j=1}^{n} b_{ij} u_j\right) u_i & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u_i(x,0) = u_{i0}(x) \geq (\not\equiv) 0 \quad \text{in } \Omega,\ i = 1,\ldots,n, \end{cases} \tag{1.2}$$

where u_i represents density of ith species with $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$, α_{ij} and a_i are all positive constants. The term positive cross-diffusion coefficient denotes that one species tends to diffuse in the direction of low concentration of another species. Each b_{ij} is a non-zero constant with the assumption

(SM): $b_{ii} > 0, \quad b_{ij} < 0$ for all $i \neq j$.

For simplicity of presentation, we assume that $\Omega = (0, 1)$.

Assumption (SM) [24] shows that problem (1.2) is an n-species cooperating system. For the time-dependent solution of (1.2), the local existence is an immediate consequence of papers [15–17]. Namely, if $u_{i0} \in W_p^1(\Omega)$ for $i = 1, \ldots, n$ and $p > N$, then (1.2) has a unique nonnegative solution $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$, where $u_i \in C\left([0, T), W_p^1(\Omega)\right) \bigcap C^\infty\left((0, T), C^\infty(\Omega)\right)$ and $T \in (0, \infty]$ is the maximal existence time of the solution. If $\mathbf{u}$ satisfies the estimates

$$\|u_i(\cdot, t)\|_{W_p^1(\Omega)} < \infty \quad \text{for all } t \in (0, T),$$

then $T = +\infty$. Further, if $u_{i0} \in W_p^2(\Omega)$, then $u_i \in C\left([0, \infty), W_p^2(\Omega)\right)$. Based on this, we will obtain the global existence of the solution by making some prior estimates for the local solution of (1.2).

This paper is organized as follows: In Section 2, we introduce the Gagliardo–Nirenberg type inequality and some corollaries. In Section 3, we investigate the uniform boundedness and global existence of the solution to (1.2) by using Amann's results. The positive definiteness of the matrix $\mathfrak{D}$(see Theorem 3.1) is indispensable to establish the L^2-estimates of u_i and $\partial_x u_i$. The L^1-estimate of u_i follows from the definiteness of the competition matrix B. As for the weakly coupled case (lacking cross-diffusions), similar to the proof of Theorem 3.1 in [4], we can employ the method of upper and lower solutions to construct a bounded positive upper solution. Then by only restricting the inter- and intra-interactions, it is not difficult to establish the global existence of a unique nonnegative solution. In Section 4, the asymptotic behavior of the positive constant steady state of (1.2) is discussed. Meanwhile, we note that the corresponding results still hold for n-species strongly coupled competitive model and prey–predator model. Although our consideration of system (1.2) is for no flux boundary condition, the case of homogeneous Dirichlet boundary conditions can be treated in the same way.

2. Preliminaries

The following Gagliardo–Nirenberg type inequality and its corollaries will play important roles in establishing the prior estimates of the solution.