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Hybrid function method for solving Fredholm and Volterra integral equations of the second kind☆

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ABSTRACT

Numerical solutions of Fredholm and Volterra integral equations of the second kind via hybrid functions, are proposed in this paper. Based upon some useful properties of hybrid functions, *integration of the cross product*, a special *product matrix* and a related *coefficient matrix* with optimal order, are applied to solve these integral equations. The main characteristic of this technique is to convert an integral equation into an algebraic; hence, the solution procedures are either reduced or simplified accordingly. The advantages of hybrid functions are that the values of n and m are adjustable as well as being able to yield more accurate numerical solutions than the piecewise constant orthogonal function, for the solutions of integral equations. We propose that the available optimal values of n and m can minimize the relative errors of the numerical solutions. The high accuracy and the wide applicability of the hybrid function approach will be demonstrated with numerical examples. The hybrid function method is superior to other piecewise constant orthogonal functions [W.F. Blyth, R.L. May, P. Widyaningsih, Volterra integral equations solved in Fredholm form using Walsh functions, Anziam J. 45 (E) (2004) C269–C282; M.H. Reihani, Z. Abadi, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, J. Comp. Appl. Math. 200 (2007) 12–20] for these problems.

1. Introduction

Integral equations are often involved in the mathematical formulation of physical phenomena. Integral equations can be encountered in various fields of science such as physics [2], biology [10] and engineering [1,7]. But we can also use it in numerous applications, such as biomechanics, control, economics, elasticity, electrical engineering, electrodynamics, electrostatics, filtration theory, fluid dynamics, game theory, heat and mass transfer, medicine, oscillation theory, plasticity, queuing theory, etc. [14]. Fredholm and Volterra integral equations of the second kind show up in studies that includes airfoil theory [9], elastic contact problems [12,16], fracture mechanics [17], combined infrared radiation and molecular conduction [8] and so on.

In recent years, many different basic functions [3,15] have been used to estimate the solution of integral equations, such as orthogonal functions and wavelets. Depending on the structure, the orthogonal functions may be widely classified into three families [6]: The first includes sets of piecewise constant orthogonal functions (PCOF) (e.g., Walsh, block-pulse, Haar, etc.). The second consists of sets of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, etc.). The third are the widely used sets of sine–cosine functions in the Fourier series.

Fredholm and Volterra integral equations of the second kind are much more difficult to solve than ordinary differential equations. Therefore, many authors [3,15] have tried various transform methods to overcome these difficulties. Recently,

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hybrid functions have been applied extensively for solving differential equations or systems, and proved to be a useful mathematical tool. The pioneering work in system analysis via hybrid functions was led in [13,18], who first derived an *operational matrix* for the integrals of the hybrid function vector, and paved the way for the hybrid function analysis of the dynamic systems. But they only derived the matrix of small order, and the calculations are not enough to achieve high accuracy.

In this paper, we present the properties of hybrid functions which consists of block-pulse functions plus Legendre polynomials. Based upon some useful properties of hybrid functions, *integration of the cross product*, a special *product matrix* and a related *coefficient matrix* with optimal order are applied to solve these integral equations. The main characteristic of this technique is to convert an integral equation into an algebraic; hence, the solution procedures are either reduced or simplified, accordingly.

Most scholars researching hybrid function, only mentioned that it can be utilized to solve the differential equations or systems. They have really neglected an important question—how large the respective ranks n and m representing the block-pulse functions and Legendre polynomials should be on earth, to yield more accurate numerical solutions. We propose that the available optimal values of n and m can minimize the relative errors of the numerical solutions.

2. Some properties of hybrid functions

2.1. Hybrid functions of block-pulse and Legendre polynomials

The orthogonal set of hybrid functions $h_{ij}(t)$, $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m-1$ is defined on the interval [0, 1) as

$$h_{ij}(t) = \begin{cases} L_j\left(2nt - 2i + 1\right), & \text{for } t \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right) \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where n and m are the order of block-pulse functions and Legendre polynomials, respectively, and t is the normalized time. $L_k(t)$ denotes the Legendre polynomials of order k satisfying

$$L_0(t) = 1, \qquad L_1(t) = t, \tag{2}$$

$$L_{k+1}(t) = \left(\frac{2k+1}{k+1}\right) t\, L_k(t) - \left(\frac{k}{k+1}\right) L_{k-1}(t), \quad k = 1, 2, 3, \ldots \tag{3}$$

2.2. Function approximation

Any function $y(t)$ which is square integrable in the interval [0, 1) can be expanded in a hybrid function

$$y(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, h_{ij}(t), \quad i = 1, 2, \ldots, \infty, j = 0, 1, \ldots, \infty, t \in [0, 1), \tag{4}$$

where the hybrid coefficients

$$c_{nm} = \frac{(y(t), h_{nm}(t))}{(h_{nm}(t), h_{nm}(t))} \tag{5}$$

are determined such that the integral square error ϵ is minimized:

$$\epsilon = \int_0^1 \left[y(t) - \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}\, h_{ij}(t) \right]^2 \mathrm{d}t. \tag{6}$$

In Eq. (5), $(. , .)$ denotes the inner product. Usually, the series expansion Eq. (4) contains an infinite number of terms for a smooth $y(t)$. If $y(t)$ is piecewise constant or may be approximated as piecewise constant, then the sum in Eq. (4) may be terminated after nm terms, that is

$$y(t) \approx \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}\, h_{ij}(t) = \mathbf{c}_{(nm)}^{\mathrm{T}}\, \mathbf{h}_{(nm)}(t) \triangleq y^*(t), \tag{7}$$

where

$$\mathbf{c}_{(nm)} \triangleq [c_{10}, \ldots, c_{1,m-1}, c_{20}, \ldots, c_{2,m-1}, \ldots, c_{n0}, \ldots, c_{n,m-1}]^{\mathrm{T}}, \tag{8}$$

and

$$\mathbf{h}_{(nm)}(t) \triangleq [h_{10}(t), \ldots, h_{1,m-1}(t), h_{20}(t), \ldots, h_{2,m-1}(t), \ldots, h_{n0}(t), \ldots, h_{n,m-1}(t)]^{\mathrm{T}}, \tag{9}$$

where the subscript nm in the parentheses denotes the vector dimensions and $y^*(t)$ denotes the truncated sum. Let us define the nm-square hybrid matrix as

$$H_{(nm\times nm)} \triangleq \left[\mathbf{h}_{(nm)}\left(\frac{1}{2nm}\right) \;\; \mathbf{h}_{(nm)}\left(\frac{3}{2nm}\right) \;\; \cdots \;\; \mathbf{h}_{(nm)}\left(\frac{2nm-1}{2nm}\right) \right]. \tag{10}$$