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A Galerkin boundary node method and its convergence analysis

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ABSTRACT

The boundary node method (BNM) exploits the dimensionality of the boundary integral equation (BIE) and the meshless attribute of the moving least-square (MLS) approximations. However, since MLS shape functions lack the property of a delta function, it is difficult to exactly satisfy boundary conditions in BNM. Besides, the system matrices of BNM are non-symmetric.

A Galerkin boundary node method (GBNM) is proposed in this paper for solving boundary value problems. In this approach, an equivalent variational form of a BIE is used for representing the governing equation, and the trial and test functions of the variational formulation are generated by the MLS approximation. As a result, boundary conditions can be implemented accurately and the system matrices are symmetric. Total details of numerical implementation and error analysis are given for a general BIE. Taking the Dirichlet problem of Laplace equation as an example, we set up a framework for error estimates of GBNM. Some numerical examples are also given to demonstrate the efficacity of the method.

1. Introduction

In recent years, the meshless (or meshfree) methods have attached much attention for solving boundary value problems [1,2]. The main feature of this type of method is the absence of an explicit mesh, and the approximate solutions are constructed entirely based on a cluster of scattered nodes. Although many types of meshless methods have been already proposed, these methods can be divided into two categories: the boundary type and the domain type. Several domain type meshless methods, such as the element free Galerkin method (EFGM) [3], the reproducing kernel particle method [4], the moving least-square reproducing kernel method [5,6], the finite point method [7] and the h–p meshless method [8] have achieved remarkable progress in solving a wide range of boundary value problems, and their mathematical backgrounds were investigated.

Boundary integral equations (BIEs) have been widely used for the solution of boundary value problems in potential theory and engineering. Based on coupling BIEs and the moving least-squares (MLS) approach [9,10], Mukherjee and Mukherjee [11] proposed a boundary type meshless method which they call the boundary node method (BNM). BNM requires only a nodal structure on the bounding surface of a body for approximation of boundary unknowns. Hence it is an attractive computational technique for linear problems compared with the domain type meshless methods. However, since the MLS approximation lacks the delta function property, BNM cannot exactly satisfy boundary conditions. And the strategy used in BNM to impose boundary conditions doubles the number of system equations. Xie et al. [12] proposed a radial boundary node method (RBNM) to overcome this difficulty by using radial basis functions instead of the MLS to construct the interpolation functions. Although RBNM has been applied to the linear elasticity problems, the accuracy of numerical results is affected by the shape parameters of radial basis functions (e.g. parameters in MQ and Gaussians basis functions [13]), and the optimal

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values of these parameters are still not determined theoretically. Moreover, as BNM, the system matrices of RBNM are non-symmetric, and the theoretical basis is just being studied and far from completion.

In this paper we present a Galerkin boundary node method (GBNM), which based on an equivalent variational form of a boundary integral formulation for the governing partial differential equation. The key ideas in GBNM are:

1. The MLS approximation is implemented to construct the trial and test functions of the variational form by a cluster of nodes instead of elements. Thus, the elements division in the boundary element method (BEM) can be avoided.
2. The 'stiffness' matrices are symmetric, which provides an added advantage in coupling GBNM with finite element method (FEM) [14] or other established meshless methods such as EFGM. This coupled technique is especially suited for the problems with an unbounded domain.
3. Although the shape functions of MLS approximation lack the delta function property, boundary conditions can be enforced by the variational formulation. Thus the implementation of boundary conditions in this method is much easier than that in other meshless methods such as in BNM or EFGM, in which the MLS is also introduced.

The rest of this paper is outlined as follows. In Section 2, we introduce some preliminaries to be used later. Section 3 gives a brief description of the MLS approximation and deduces its error estimates. Then, a detailed numerical implementation of GBNM is described and the theoretical analysis of this method in Sobolev spaces is provided in the next section. Section 5 provides some numerical tests on theoretical results of the proposed meshless method. Finally, the conclusion is presented in Section 6.

2. Preliminary

Let Ω be an open bounded domain in $\mathbb{R}^2$ with boundary Γ, the complement of $\bar{\Omega} = \Omega + \Gamma$ is denoted by Ω'. A generic point in $\mathbb{R}^2$ is denoted by $\mathbf{x} = (x_1, x_2)$ or $\mathbf{y} = (y_1, y_2)$.

For any $\mathbf{x} \in \Gamma$, assume that the influence domain of $\mathbf{x}$ is $\Re(\mathbf{x})$ with radius $r(\mathbf{x})$, then $\Re(\mathbf{x})$ is a piece of the boundary and can be represented by a curvilinear co-ordinate (here the arc length) s, i.e.,

$$\Re(\mathbf{x}(s)) := \left\{\mathbf{y}\left(\tilde{s}\right) \in \Gamma : \left|\tilde{s} - s\right| \le r(\mathbf{x})\right\}, \tag{1}$$

where $\tilde{s}$ is the curvilinear coordinate of the boundary point $\mathbf{y}$.

Obviously, if Γ is a C^{ℓ_Γ} curve, it is true that $\Re(\mathbf{x})$ is a C^{ℓ_Γ} curve, thus $\partial^m \mathbf{x}(s)/\partial s^m$ is bounded provided that $m \le \ell_\Gamma$.

Let $\mathbf{x}_i \in \Gamma$ $(1 \le i \le N)$ be a set of points which are called boundary nodes. On $\Re(\mathbf{x})$, the curvilinear co-ordinate of $\mathbf{x}_i \in \Re(\mathbf{x})$ is denoted by s_i. Besides, assume that there have $\kappa(\mathbf{x})$ boundary nodes that lie on $\Re(\mathbf{x})$. Then, we use the notation $I_1, I_2, \ldots, I_\kappa$ to express the global sequence number of these nodes, and define $\wedge(\mathbf{x}) := \{I_1, I_2, \ldots, I_\kappa\}$.

From (1) the influence domain of $\mathbf{x}_i$ is

$$\Re_i := \Re(\mathbf{x}_i(s)) = \left\{\mathbf{y}\left(\tilde{s}\right) \in \Gamma : \left|\tilde{s} - s\right| \le r(\mathbf{x}_i)\right\}, \quad 1 \le i \le N. \tag{2}$$

It is worth noting that the union of $\{\Re_i\}_{i=1}^N$ should be a finite open covering of Γ, i.e., $\Gamma \subset \bigcup_{i=1}^N \Re_i$.

Besides, we use

$$\Re^i := \{\mathbf{x} \in \Gamma : \mathbf{x}_i \in \Re(\mathbf{x})\}, \quad 1 \le i \le N, \tag{3}$$

to denote the set of boundary points whose influence domain including the boundary node $\mathbf{x}_i$. For a different boundary point $\mathbf{x}$, the influence domain $\Re(\mathbf{x})$ varies from point to point, hence $\Re^i \equiv \Re_i$ if and only if $r(\mathbf{x})$ is a constant for any $\mathbf{x} \in \Gamma$.

For convenience, we suppose that τ is real and we denote by $H^\tau(\Gamma)$ the Sobolev spaces as well as their interpolation spaces on Γ for noninteger τ [15]. Moreover, let m be a nonnegative integer, we define the following weighted Sobolev spaces [16]

$$W^m_{m-1}\left(\Omega'\right) := \left\{u \in \mathscr{D}'\left(\Omega'\right) : \frac{u}{\sqrt{1+r^2}\ln\left(2+r^2\right)} \in L^2\left(\Omega'\right), \left(1+r^2\right)^{(|\lambda|-1)/2} D^\lambda u \in L^2\left(\Omega'\right), 1 \le |\lambda| \le m\right\},$$

where $\lambda = (\lambda_1, \lambda_2)$, $|\lambda| = \lambda_1 + \lambda_2$, and $r = |\mathbf{x}|$ represents the distance from the origin to the point $\mathbf{x} \in \mathbb{R}^2$.

The norm in $W^m_{m-1}\left(\Omega'\right)$ is defined by

$$\|u\|_{W^m_{m-1}(\Omega')} := \left(\left\|\frac{u}{\sqrt{1+r^2}\ln\left(2+r^2\right)}\right\|^2_{L^2(\Omega')} + \sum_{|\lambda|=1}^{m} \left\|\left(1+r^2\right)^{(|\lambda|-1)/2} D^\lambda u\right\|^2_{L^2(\Omega')}\right)^{\frac{1}{2}}.$$

Observe that all the local properties of the space $W^m_{m-1}(\Omega')$ coincide with those of the Sobolev space $H^m(\Omega')$. As a consequence, the traces of these functions on Γ satisfy the usual trace theorems.

3. The moving least squares (MLS) method

The MLS as an approximation method has been introduced in [9,10]. Since the numerical approximations of MLS starting from a cluster of scattered nodes instead of interpolation on elements, there have many meshless methods based on the MLS method for the numerical solution of differential equations in recent years.