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Pseudo-differential operators associated with Laguerre hypergroups

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ABSTRACT

In this paper we consider the generalized shift operator generated from the Laguerre hypergroup; by means of this, pseudo-differential operators are investigated and Sobolev-boundedness results are obtained.

1. Introduction

Throughout this paper we fix $\alpha \geq 0$ and we use the notation

$$\varphi_{\lambda,j}(x,t) = e^{i\lambda t} e^{-\frac{|\lambda|x^2}{2}} L_j^\alpha(|\lambda|x^2)/L_j^\alpha(0),$$

for all $(x,t) \in \mathbb{K} = [0,\infty[\,\times\mathbb{R}$ and $(\lambda,j) \in \widehat{\mathbb{K}} = \mathbb{R}\setminus\{0\}\times\mathbb{N}$, where L_j^α is the Laguerre polynomial of degree j and order α. Using the product formula for the Laguerre functions proved by Koorwinder in 1977 (see [1], p. 537), Nessibi and Trimèche in [2] proved that the functions $\varphi_{(\lambda,j)}$ introduced above satisfy the following product formula:

$$\varphi_{\lambda,j}(x,t)\varphi_{\lambda,j}(y,s) = \int_{\mathbb{K}} \varphi_{\lambda,j} \mathrm{d}w_\alpha \tag{1}$$

where $\mathrm{d}w_\alpha$ is a probability measure on $\mathbb{K}$. For $\xi = (x,t)$, $\eta = (y,z) \in \mathbb{K}$, $\theta \in [0,2\pi[$ and $r \in [0,1]$, put

$$[\xi,\eta]_{\theta,r} = (\sqrt{x^2+y^2+2rxy\cos\theta},\, t+z+xyr\sin\theta).$$

Then the formula (1) can be expressed for $\alpha > 0$ as follows:

$$\varphi_{\lambda,j}(\xi)\varphi_{\lambda,j}(\eta) = \int_{[0,1]\times[0,2\pi]} \varphi_{\lambda,j}([\xi,\eta]_{\theta,r}) r(1-r^2)^{\alpha-1} \mathrm{d}r\mathrm{d}\theta. \tag{2}$$

The product formula above generates a shifted translation operator given for a suitable function f by

$$\tau_\xi f(\eta) = \int_{[0,1]\times[0,2\pi]} f([\xi,\eta]_{\theta,r}) r(1-r^2)^{\alpha-1} \mathrm{d}r\mathrm{d}\theta \tag{3}$$

which can be expressed differently with a suitable kernel (see [3], p. 122). These translations allow one to introduce a generalized hypergroup in the sense of Jewett [4] with Haar measure $\mathrm{d}m_\alpha(x,t) = \frac{x^{2\alpha+1}\mathrm{d}x\mathrm{d}t}{\pi\Gamma(\alpha+1)}$ and involution "$^-$" given by: $\xi^- = (x,-t)$ for all $\xi \in \mathbb{K}$, called the Laguerre hypergroup (see [2]). The convolution product for an appropriate pair of

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functions f and g is given by

$$f\star g(\xi)=\int_{\mathbb{K}}\tau_{\xi}f(\eta)g(\eta^{-})\mathrm{d}m_{\alpha}(\eta), \tag{4}$$

while the Fourier–Laguerre transform of a suitable function f on $\mathbb{K}$ is given by

$$\widehat{f}(\lambda,j)=\int_{\mathbb{K}}\varphi_{-\lambda,j}f\,\mathrm{d}m_{\alpha}.$$

The case $\alpha=0$ can be treated in a similar way.

The functional analysis and Fourier analysis on $\mathbb{K}$ and its dual have been extensively studied in [5,2], and hence it is well known that the Fourier–Laguerre transform given above is a topological isomorphism from the Schwartz space on $\mathbb{K}$ onto $S(\widehat{\mathbb{K}})$: the Schwartz space on $\widehat{\mathbb{K}}$ (see [2, Proposition II.1]). Its inverse is given by

$$g^{\vee}(\xi)=\int_{\widehat{\mathbb{K}}}\varphi_{.}(\xi)g\,\mathrm{d}\gamma_{\alpha} \tag{5}$$

where $\mathrm{d}\gamma$ is the Plancherel measure on $\widehat{\mathbb{K}}$ given by $\mathrm{d}\gamma(\lambda,j)=|\lambda|^{\alpha+1}L_j^{\alpha}(0)\delta_j\bigotimes\mathrm{d}\lambda$.

The topology on $\mathbb{K}$ is given by the norm $N(x,t)=(x^4+4t^2)^{\frac{1}{4}}$, while we assign to $\widehat{\mathbb{K}}$ the topology generated by the quasi-semi-norm $\mathcal{N}(\lambda,j)=4|\lambda|(j+\frac{\alpha+1}{2})$. In fact, the dual of the Laguerre hypergroup $\widehat{\mathbb{K}}$ can be topologically identified with the so-called Heisenberg fan [6], i.e., the subset embedded in $\mathbb{R}^2$ given by

$$\Big(\bigcup_{j\in\mathbb{N}}\{(\lambda,\mu)\in\mathbb{R}^2:\mu=|\lambda|(2j+\alpha+1),\lambda\neq 0\}\Big)\bigcup\{(0,\mu)\in\mathbb{R}^2:\mu\geq 0\}.$$

Moreover, the subset $\{(0,\mu)\in\mathbb{R}^2:\mu\geq 0\}$ has zero Plancherel measure; therefore it will usually be disregarded.

In the present work, we introduce pseudo-differential operators in the setting of the Laguerre hypergroups and we study Sobolev-boundedness results. Although an in-depth study of hypergroups falls outside the scope of this paper, it seems appropriate to collect some harmonic analysis results in such a setting, which is close to that of the hypergroup of radial functions on the Heisenberg group (see for example [5,3,2]). Hence, we recall some notions which turned out to be relevant for different purposes and we focus here on the support of the probability measure $\mathrm{d}w_{\alpha}$ in order to give the main technical tool (see Lemma 1).

This paper is organized as follows. In the second section we collect some standard Fourier analysis results on the Laguerre hypergroup that will be used in the sequel (see [3,2]). The third section is devoted to introducing generalized pseudo-differential operators on the Laguerre hypergroup and to proving a Sobolev-boundedness theorem.

Throughout this paper, C will always represent a positive constant, not necessarily the same in each occurrence.

2. Preliminaries

To describe the harmonic analysis in our setting we begin with introducing the operator $\mathcal{L}_{\alpha}=\frac{\partial^2}{\partial x^2}+\frac{2\alpha+1}{x}\frac{\partial}{\partial x}+x^2\frac{\partial^2}{\partial t^2}$ defined on $\mathbb{K}$, and also the operator $\Lambda=\left(\Lambda_1^2-(2\Lambda_2+2\frac{\partial}{\partial\lambda})^2\right)$ defined on $\widehat{\mathbb{K}}$, where $\Lambda_1=\frac{1}{|\lambda|}\left(j\Delta_+\Delta_-+(\alpha+1)\Delta_+\right)$ and $\Lambda_2=\frac{-1}{2\lambda}\left((\alpha+j+1)\Delta_++j\Delta_-\right)$.

$\Delta_{\pm}$ are given for a suitable function Φ by: $\Delta_+\Phi(\lambda,j)=\Phi(\lambda,j+1)-\Phi(\lambda,j)$, $\Delta_-\Phi(\lambda,j)=\Phi(\lambda,j)-\Phi(\lambda,j-1)$, if $j\geq 1$ and $\Delta_-\Phi(\lambda,0)=\Phi(\lambda,0)$.

These operators generate the harmonic analysis on the Laguerre hypergroup and its dual which can be found in [3,5,2]; namely one has

$$\mathcal{L}_{\alpha}\varphi_{\lambda,j}=-\mathcal{N}\lambda,j\varphi_{\lambda,j}\quad\text{and}\quad\Lambda\varphi_{\lambda,j}(\xi)=N^4(\xi)\varphi_{\lambda,j}. \tag{6}$$

Moreover, for an appropriate Φ it follows (see [5]) that

$$[(1+\Lambda)\Phi]^{\vee}(\xi)=[1+N^4(\xi)]\Phi^{\vee}(\xi). \tag{7}$$

The Sobolev spaces $H_p^s(\widehat{\mathbb{K}})$ introduced here (see [5]) are defined to be the collection of tempered distributions f on $\widehat{\mathbb{K}}$ whose inverse Fourier transform $f^{\vee}$ belongs to $L^p(\mathbb{K}):=L^p(\mathbb{K},\mathrm{d}m_{\alpha})$, and satisfies

$$\|f\|_{H_p^s}:=\left\|(1+N^4)^sf^{\vee}\right\|_{L^p(\mathbb{K})}<\infty.$$

A symbol σ in the Hormander class $S^{1,m}$ ($m\in\mathbb{N}$) will be a $\mathcal{C}^{\infty}$ function of $\widehat{\mathbb{K}}\times\mathbb{K}$ satisfying

$$\sup_{\widehat{\mathbb{K}}\times\mathbb{K}}\left\{\frac{(1+\mathcal{N}(\lambda,j))^{\delta}}{(1+N^4(\xi))^{m-\gamma}}\Lambda^{\beta}\mathcal{L}_{\alpha}^{\gamma}\sigma((\lambda,j),\xi)\right\}<\infty,\quad\text{for all }\beta,\gamma,\delta\in\mathbb{N}. \tag{8}$$

Moreover, the following proposition holds.

Proposition 1. *For all σ in $S^{1,m}$ and for all $k\in\mathbb{N}$ there exists a constant $C_{m,k}>0$ such that*

$$\left|[\sigma^{\vee}(.,\xi)](\eta)\right|\leq C_{m,k}(1+N^4(\xi))^m(1+N^4(\eta))^{-k}. \tag{9}$$