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Asymptotic analysis of the Bell polynomials by the ray method

Diego Dominici

Department of Mathematics, State University of New York at New Paltz, 1 Hawk Dr., New Paltz, NY 12561-2443, USA

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ABSTRACT

We analyze the Bell polynomials $B_n(x)$ asymptotically as $n \to \infty$. We obtain asymptotic approximations from the differential-difference equation which they satisfy, using a discrete version of the ray method. We give some examples showing the accuracy of our formulas.

1. Introduction

The Bell polynomials $B_n(x)$ are defined by [1]

$$B_n(x) = \sum_{k=0}^{n} S_k^n x^k, \quad n = 0, 1, \dots,$$

where S_k^n is a Stirling number of the second kind [2]. They have the generating function

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \exp\left[x\left(e^t - 1\right)\right], \tag{1}$$

from which it follows that

$$B_0(x) = 1 \tag{2}$$

and

$$B_{n+1}(x) = x\left[B_n'(x) + B_n(x)\right], \quad n = 0, 1, \dots. \tag{3}$$

The first few $B_n(x)$ are

$$B_1(x) = x, \qquad B_2(x) = x(1+x), \qquad B_3(x) = x\left(1 + 3x + x^2\right),$$
$$B_4(x) = x\left(1 + 7x + 6x^2 + x^3\right), \qquad B_5(x) = x\left(1 + 15x + 25x^2 + 10x^3 + x^4\right)$$

and in general

$$B_n(x) = x + \left(2^{n-1} - 1\right)x^2 + \cdots + \frac{n(n-1)}{2}x^{n-1} + x^n. \tag{4}$$

E-mail address: dominicd@newpaltz.edu.

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The arithmetic properties of $B_n(x)$ were considered in [3] and in [4], where he also showed that the polynomials $B_n(x)$ have simple roots. The same result was proved in [5], using other arguments.

The strong asymptotics of $B_n(x)$ [6] and their zero distribution [7] was studied by C. Elbert applying the saddle point method to (1). Zhao [8], investigated the uniform asymptotic behavior of $B_n(x)$ on the negative real axis, where all the zeros are located. He obtained an expansion in terms of the Airy function and its derivative.

In this paper, we will use a different approach and analyze (3) instead of (1). The advantage of our method is that no knowledge of a generating function is required and therefore it can be applied to other sequences of polynomials satisfying differential-difference equations [9,10].

2. Asymptotic analysis

To analyze (3) asymptotically as $n \to \infty$, we use a discrete version of the ray method [11]. Replacing the ansatz

$$B_n(x) = \varepsilon^{-n} F(\varepsilon x, \varepsilon n) \tag{5}$$

in (3), we get

$$F(u, v+\varepsilon) = u\left(\varepsilon \frac{\partial F}{\partial x} + F\right), \tag{6}$$

with

$$u = \varepsilon x, \qquad v = \varepsilon n \tag{7}$$

and ε is a small parameter. We consider asymptotic solutions for (6) of the form

$$F(u, v) \sim \exp\left[\varepsilon^{-1}\psi(u, v)\right] K(u, v), \tag{8}$$

as $\varepsilon \to 0$. Using (8) in (6) we obtain, to leading order, the eikonal equation

$$\mathrm{e}^{q} - u(p+1) = 0 \tag{9}$$

and the transport equation

$$\frac{\partial K}{\partial v} + \frac{1}{2}\frac{\partial^2 \psi}{\partial v^2} K - u \exp\left(-\frac{\partial \psi}{\partial v}\right)\frac{\partial K}{\partial u} = 0, \tag{10}$$

where

$$p = \frac{\partial \psi}{\partial x}, \qquad q = \frac{\partial \psi}{\partial v}. \tag{11}$$

The initial condition (2), implies

$$\psi(u, 0) = 0, \qquad K(u, 0) = 1. \tag{12}$$

To solve (9) we use the method of characteristics, which we briefly review. Given the first order partial differential equation

$$\mathfrak{F}(u, v, \psi, p, q) = 0,$$

with p, q defined in (11), we search for a solution $\psi(u, v)$ by solving the system of "characteristic equations"

$$\begin{aligned}
&\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial \mathfrak{F}}{\partial p}, \qquad \dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial \mathfrak{F}}{\partial q},\\
&\dot{p} = \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial \mathfrak{F}}{\partial u} - p\frac{\partial \mathfrak{F}}{\partial \psi}, \qquad \dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial \mathfrak{F}}{\partial v} - q\frac{\partial \mathfrak{F}}{\partial \psi},\\
&\dot{\psi} = \frac{\mathrm{d}\psi}{\mathrm{d}t} = p\frac{\partial \mathfrak{F}}{\partial p} + q\frac{\partial \mathfrak{F}}{\partial q},
\end{aligned}$$

where we now consider $\{u, v, \psi, p, q\}$ to all be functions of the new variables t and s.

For (9), we have

$$\mathfrak{F}(u, v, \psi, p, q) = \mathrm{e}^{q} - u(p+1)$$

and therefore the characteristic equations are

$$\dot{u} + u = 0, \qquad \dot{v} = \mathrm{e}^{q}, \qquad \dot{p} - p = 1, \qquad \dot{q} = 0. \tag{13}$$