Journal of Computational and Applied Mathematics 233 (2009) 762–767

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convexity of the zeros of some orthogonal polynomials and related functions

Kerstin Jordaan [a,∗], Ferenc Tookos [b]

[a] *Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa*

[b] *Institute for Biomathematics and Biometry, GSF - Research Center for Environment and Health, Neuherberg, Germany*

ARTICLE INFO

Article history:
Received 29 October 2007

MSC:
33C45
34C10
42C05

Keywords:
Orthogonal polynomials
Zeros
Laguerre polynomials
Jacobi polynomials
Ultraspherical polynomials
Convexity

ABSTRACT

We study convexity properties of the zeros of some special functions that follow from the convexity theorem of Sturm. We prove results on the intervals of convexity for the zeros of Laguerre, Jacobi and ultraspherical polynomials, as well as functions related to them, using transformations under which the zeros remain unchanged. We give upper as well as lower bounds for the distance between consecutive zeros in several cases.

1. Introduction

The Sturm comparison theorem for solutions of second-order differential equations of the form $y'' + F(t)y = 0$ (cf. [1]) has been significantly extended since its publication 170 years ago. Some of the immediate applications to zeros of the solutions $y(t)$, and those of the derivative $y'(t)$, include Sonin's theorem on the monotonicity of extrema of such solutions (cf. [2]), and a result known as Sturm's convexity theorem, first mentioned in [1], on the monotonicity of distances between the zeros of the solution (cf. [3,2,4]).

Sonin's theorem was extended to more general differential equations of the form $P(t)y'' + Q(t)y' + y = 0$ using a remarkably simple proof (cf. [5], p. 443). In this form the theorem can be directly applied to classical orthogonal polynomials such as Hermite, Laguerre and Jacobi polynomials, providing the monotonicity of their relative maximum values and estimates on their supremum norm. This has been done from a different perspective in [6], recovering results for Legendre, Laguerre and Jacobi polynomials given in [7].

In this paper, we consider the implications of the convexity theorem of Sturm for the convexity of the zeros and bounds on the distance between the zeros of some classical orthogonal polynomials and functions related to them.

2. Convexity and spacing of zeros

The convexity theorem and an obvious consequence of the comparison theorem of Sturm, already noted in [1], can be summarised as follows.

∗ Corresponding author.
E-mail address: kjordaan@up.ac.za (K. Jordaan).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.02.045

Theorem 2.1 ([8]). *Let $y''(t) + F(t)y(t) = 0$ be a second-order differential equation in normal form, where F is continuous in (a, b). Let $y(t)$ be a nontrivial solution in (a, b), and let $x_1 < \cdots < x_k < x_{k+1} < \cdots$ denote the consecutive zeros of $y(t)$ in (a, b). Then*

1. *if $F(t)$ is strictly increasing in (a, b), $x_{k+2} - x_{k+1} < x_{k+1} - x_k$,*
2. *if $F(t)$ is strictly decreasing in (a, b), $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.*
3. *if there exists $M > 0$ such that $F(t) < M$ in (a, b) then*
$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{M}},$$
4. *if there exists $m > 0$ such that $F(t) > m$ in (a, b) then*
$$\Delta x_k < \frac{\pi}{\sqrt{m}}.$$

We say that the zeros of y are concave (convex) on (a, b) for the first (second) case.

The convexity theorem has been used to obtain the variation of convexity properties with respect to a parameter, or the order, for the zeros of gamma, q-gamma, Bessel, cylindrical and Hermite functions as described in the survey paper [9].

In order to apply the convexity theorem to special functions that are solutions of second-order differential equations, the differential equation has to be transformed into normal form. One simple way to do this is through the following change of dependent variable. Let

$$x'' + g(t)x' + f(t)x = 0$$

be a second-order differential equation and set

$$y = x \exp\left(\frac{1}{2}\int^t g(s)\mathrm{d}s\right). \tag{1}$$

The corresponding equation for y is in normal form:

$$y'' + F(t)y = 0,$$

where $F(t) = f(t) - \frac{1}{4}g^2(t) - \frac{1}{2}g'(t)$. The advantage of this transformation is that it does not change the independent variable, and the zeros of x and y are the same. Hille [10] already used transformation (1) to prove the convexity of zeros of the Hermite polynomials.

It is also possible to consider other changes of variable and obtain information on the convexity of the transformed zeros. This was done already by Szegő for the ultraspherical polynomials [7, Theorem 6.3.3] and lately by Deaño, Gil and Segura [8,11] for hypergeometric functions.

We will consider the convexity and spacing of the zeros of special functions such as Laguerre, Jacobi and, as a special case, the ultraspherical polynomials, for fixed order n, by transforming their differential equations to normal form using (1). Sturm [1] used the same method to obtain results on the convexity and spacing of the zeros of the Bessel function. Interesting work on the spacing of the zeros of Jacobi polynomials, as the degree changes, is done in [12]. For higher monotonicity refer to, amongst others, [13,14].

We note that since the convexity theorem is applicable to any oscillating solutions of second-order differential equations in normal form, the results we obtain are not restricted to the polynomial cases, i.e. n need not necessarily be an integer, as long as the corresponding functions are oscillating on the interval under consideration. In addition, the results can be extended to parameter values where the polynomials are no longer orthogonal, since quasi-orthogonality ensures the existence of some zeros on the interval of orthogonality (cf. [15]).

3. Laguerre polynomials

The differential equation

$$tx'' + (\alpha + 1 - t)x' + nx = 0$$

satisfied by the Laguerre polynomials, $L_n^{\alpha}(t)$, orthogonal on $(0, \infty)$ with respect to the weight function $t^{\alpha}e^{-t}$ when $\alpha > -1$, is transformed to

$$y'' + F(t)y = 0$$

by (1) where

$$F(t) = \frac{-t^2 + 2\alpha t + 2t + 4nt - \alpha^2 + 1}{4t^2}. \tag{2}$$

$F'(t)$ changes sign at

$$t_0 := \frac{\alpha^2 - 1}{\alpha + 2n + 1}. \tag{3}$$