Journal of Computational and Applied Mathematics 233 (2009) 852–855

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A formula for inserting point masses☆

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ARTICLE INFO

Article history:
Received 27 October 2007

MSC:
42C05
30E10
05E35

Keywords:
Point masses
Decay of Verblunsky coefficients

ABSTRACT

Let $d\mu$ be a probability measure on the unit circle and $d\nu$ be the measure formed by adding a pure point to $d\mu$. We give a formula for the Verblunsky coefficients of $d\nu$, based on a result of Simon.

1. Introduction

Suppose we have a probability measure $d\mu$ on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. We define the inner product associated with $d\mu$ and the norm on $L^2(\partial\mathbb{D}, d\mu)$ respectively by

$$\langle f, g\rangle = \int_{\partial\mathbb{D}} \overline{f(e^{i\theta})} g(e^{i\theta}) d\mu(\theta) \tag{1.1}$$

$$\|f\|_{d\mu} = \left(\int_{\partial\mathbb{D}} |f(e^{i\theta})|^2 d\mu(\theta)\right)^{1/2}. \tag{1.2}$$

The family of monic orthogonal polynomials associated with the measure $d\mu$ is denoted as $(\Phi_n(z, d\mu))_{n=0}^{\infty}$, while the normalized family is denoted as $(\varphi_n(z, d\mu))_{n=0}^{\infty}$.

Let $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$ and $\varphi_n^*(z) = \Phi_n^*(z)/\|\Phi_n\|$ be the reversed polynomials. Orthogonal polynomials obey the Szegő recursion relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \tag{1.3}$$

α_n is called the nth Verblunsky coefficient. It is well known that there is a one-to-one correspondence between $d\mu$ and $(\alpha_j(d\mu))_{j=0}^{\infty}$ and that the Verblunsky coefficients carry much information about the family of orthogonal polynomials. For example,

$$\|\Phi_n\|^2 = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2). \tag{1.4}$$

For a comprehensive introduction to the theory of orthogonal polynomials on the unit circle, the reader should refer to [1,2], or the classic reference [3].

The result that we would like to present is the following

☆ Supported by the Croucher Foundation, Hong Kong
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doi:10.1016/j.cam.2009.02.052

Theorem 1.1. *Suppose* $\mathrm{d}\mu$ *is a probability measure on the unit circle and* $0 < \gamma < 1$. *Let* $\mathrm{d}\nu$ *be the probability measure formed by adding a point mass* $\zeta = \mathrm{e}^{\mathrm{i}\omega} \in \partial\mathbb{D}$ *to* $\mathrm{d}\mu$ *in the following manner*

$$\mathrm{d}\nu = (1-\gamma)\mathrm{d}\mu + \gamma\delta_{\omega}. \tag{1.5}$$

Then the Verblunsky coefficients of $\mathrm{d}\nu$ *are given by*

$$\alpha_n(\mathrm{d}\nu) = \alpha_n + \frac{(1-|\alpha_n|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(\zeta)}\overline{\varphi_{n+1}(\zeta)}\varphi_n^*(\zeta) \tag{1.6}$$

where

$$K_n(\zeta) = \sum_{j=0}^{n} |\varphi_j(\zeta)|^2 \tag{1.7}$$

and all objects without the label $(\mathrm{d}\nu)$ *are associated with the measure* $\mathrm{d}\mu$.

The proof is based on a result obtained by Simon in the proof of Theorem 10.13.7 in [2] (See Theorem 2.1 below). In fact, the following formula had been found in [4]

$$\Phi_n(z, \mathrm{d}\nu) = \Phi_n(z) - \frac{\Phi_n(\zeta)K_{n-1}(z,\zeta)}{(1-\gamma)\gamma^{-1} + K_{n-1}(\zeta,\zeta)}. \tag{1.8}$$

The formula for the real case was rediscovered in [5], Later, the same formula for the unit circle case was rediscovered in [6]. Unaware of Geronimus' result and the fact that Nevai's result also applies to the unit circle, Simon reconsidered this problem and proved formula (2.7) independently using a totally different method.

For applications of formula (1.6), the reader may refer to [7,8].

2. The proof

First, we will prove a few lemmas.

Lemma 2.1. *Let* $\beta_{jk} = \langle \Phi_j(\mathrm{d}\mu), \Phi_k(\mathrm{d}\mu)\rangle_{\mathrm{d}\nu}$. *Then*

$$\Phi_n(\mathrm{d}\nu)(z) = \frac{1}{D^{(n-1)}}\begin{vmatrix} \beta_{00} & \beta_{0\,1} & \cdots & \beta_{0\,n} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \cdots & \beta_{n-1\,n} \\ \Phi_0(\mathrm{d}\mu) & \cdots & \cdots & \Phi_n(\mathrm{d}\mu) \end{vmatrix} \tag{2.1}$$

where

$$D^{(n-1)} = \begin{vmatrix} \beta_{0\,0} & \beta_{0\,1} & \cdots & \beta_{0\,n-1} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \cdots & \beta_{n-1\,n-1} \end{vmatrix}. \tag{2.2}$$

Proof. Let $\tilde{\Phi}_n(\mathrm{d}\nu)$ be the right-hand side of (2.1). We observe that the inner product $\langle \Phi_j(\mathrm{d}\mu), \tilde{\Phi}_n(\mathrm{d}\nu)\rangle_{\mathrm{d}\nu}$ is zero for $j = 0, 1, \ldots, n-1$ as the last row and the jth row of the determinant are the same. By expanding in minors, we see that the leading coefficient of $\tilde{\Phi}_n(\mathrm{d}\nu)$ in (2.1) is one. In other words, $\tilde{\Phi}_n(\mathrm{d}\nu)$ is an nth degree monic polynomial which is orthogonal to $1, z, \ldots, z^{n-1}$ with respect to $\langle\, ,\, \rangle_{\mathrm{d}\nu}$, hence $\tilde{\Phi}_n(\mathrm{d}\nu)$ equals $\Phi_n(\mathrm{d}\nu)$. □

Lemma 2.2. *Let C be the following* $(n+1) \times (n+1)$ *matrix*

$$\begin{pmatrix} A & v \\ w & \beta \end{pmatrix} \tag{2.3}$$

where A is an $n \times n$ *matrix,* β *is in* $\mathbb{C}$, v *is the column vector* $(v_0, v_1, \ldots, v_{n-1})^{\mathrm{T}}$ *and* w *is the row vector* $(w_0, w_1, \ldots, w_{n-1})$. *If* $\det(A) \neq 0$, *we have*

$$\det(C) = \det(A)\left(\beta - \sum_{0\le j,k\le n-1} w_k v_j (A^{-1})_{jk}\right). \tag{2.4}$$