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# Oscillation of even order partial differential equations with distributed deviating arguments

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ABSTRACT

Some Kamenev-type oscillation criteria are established for a class of boundary value problems associated with even-order partial differential equations with distributed deviating arguments. Our approach is to reduce the high-dimensional oscillation problem to a one-dimensional oscillation one, and the general means developed by Philos and Wong is used as the main tool. The results obtained here extend and improve some known results in the literature.



## 1. Introduction and preliminaries

In this paper, we consider the following even order neutral-type partial differential equation with distributed deviating arguments.

$$\begin{aligned}&\frac{\partial}{\partial t}\left(r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}[u(x,t)+c(t)u(x,t-\tau)]\right)\\&\quad=a_0(t)\Delta u(x,t)+a_1(t)\Delta u(x,t-\nu)-\int_a^b F[t,\xi,u(x,g(t,\xi))]\mathrm{d}\mu(\xi),\qquad(1.1)\\&(x,t)\in\Omega\times\mathbb{R}_0\equiv G,\end{aligned}$$

subject to the following boundary condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_0,\qquad(1.2)$$

where $\tau$ and $\nu$ are positive constants, $m$ is an even positive integer, $\Delta$ is the Laplacian operator in $\mathbb{R}^n$, $\mathbb{R}_+=(0,\infty)$, $\mathbb{R}_0=[0,\infty)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$. The integral of (1.1) is a Stieltjes one.

Throughout this paper, we assume that the following conditions hold.

(A1) $a_0(t), a_1(t), c(t)\in \mathbf{C}(\mathbb{R}_0,\mathbb{R}_0)$, $0\le c(t)\le 1$;

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(A2) $g(t,\xi) \in \mathbf{C}(\mathbb{R}_0 \times [a,b], \mathbb{R})$ is nondecreasing with respect to $t$ and $\xi$, and $g(t,\xi) \leq t$ for $\xi \in [a,b]$, and $\liminf_{t\to\infty,\xi\in[a,b]} g(t,\xi) = \infty$, $(\mathrm{d}/\mathrm{d}t)g(t,a)$ exists;
(A3) $\mu(\xi) \in \mathbf{C}([a,b], \mathbb{R})$ is nondecreasing;
(A4) For $F \in \mathbf{C}([t_0,\infty) \times \mathbb{R}, \mathbb{R})$, there exist functions $q_m(t,\xi) \in \mathbf{C}([t_0,\infty) \times [a,b], \mathbb{R}_+)$, and $q_m(t,\xi)$ is not identically zero for all large $t$, $\sigma \in \mathbf{C}([t_0,\infty), \mathbb{R}_+)$, and a constant $p \geq 1$ such that

$$F(t,\xi,u)\operatorname{sign} u \geq q_m(t,\xi)|u|^p \quad \text{for } u \neq 0 \text{ and } t \geq t_0,$$

and

$$\sigma(t) \leq \min\{t, g(t,a)\}, \qquad \sigma'(t) > 0 \quad \text{for } t \geq t_0, \quad \text{and} \quad \lim_{t\to\infty} \sigma(t) = \infty.$$

(A5) $r \in \mathbf{C}^1([t_0,\infty), \mathbb{R}_+)$, $\lim_{t\to\infty} \int_{t_0}^t r^{-1}(s)\mathrm{d}s = \infty$, $\liminf_{t\to\infty} r(t) = c > 0$. For any $\varepsilon > 0$, there exists a $t_\varepsilon > t_0$, such that $|r'(t)| \leq \varepsilon Q(t)$ for all $t \geq t_\varepsilon$,
where

$$Q(t) = \left(\int_\Omega \phi(x)\mathrm{d}x\right)^{1-p} \int_a^b q_m(t,\xi)[1 - c(g(t,\xi))]^p \mathrm{d}\mu(\xi),$$

and the function $\phi(x) > 0$ in $x \in \Omega$ is the corresponding eigenfunction of the Dirichlet problem (2.1) and (2.2) given in Section 2.

It is well-known that partial functional differential equations (PFDE) arise from many biological, chemical, and physical systems which are characterized by both spatial and temporal variables and exhibit various spatio-temporal patterns [1,12]. In past years, the fundamental theory of PFDE has been investigated extensively by many scholars. We refer the reader to the monograph [12]. On the other hand, we notice that the oscillation theory for high-order PFDE is an object of long standing interest [1,5,6,10]. However, to the best of our knowledge, there are very few results dealing with the oscillation of the solutions of Eq. (1.1) in general form, although Wang et al. [10] have obtained some oscillation criteria for boundary value problems of even order linear PFDE

$$\frac{\partial^m}{\partial t^m}[u(x,t) + c(t)u(x,t-\tau)] = a_0(t)\Delta u(x,t) + a_1(t)\Delta u(x,t-\nu) - \int_a^b q(x,t,\xi)u(x,g(t,\xi))\mathrm{d}\mu(\xi), \tag{1.3}$$

$$(x,t) \in \Omega \times \mathbb{R}_0 \equiv G.$$

Obviously, (1.3) is a special form of Eq. (1.1).

The objective of this paper is to establish Kamenev-type oscillation criteria [7] of solutions to the boundary value problem given by (1.1) and (1.2). Our approach is to reduce the high-dimensional oscillation problem to a one-dimensional oscillation one, and the general means developed by Philos and Wong is used as the main tool. The results obtained here extend and improve the main results in [10].

For completeness, we first introduce the following concepts and Lemmas.

**Definition 1.1.** A function $u \in \mathbf{C}^2(G) \cap \mathbf{C}^1(\overline{G})$ is said to be a solution of the boundary value problem (1.1) and (1.2), if it satisfies (1.1) in the domain $G$ and the boundary condition on the boundary.

**Definition 1.2.** A solution $u(x,t)$ of the boundary value problem (1.1) and (1.2) is said to be oscillatory in the domain $G$, if for any positive number $t_\mu$ there exists a point $(x_0,t_0) \in \Omega \times [t_\mu,\infty)$ such that the condition $u(x_0,t_0) = 0$ holds.

Next, we introduce the general means developed in [8,11], and present some properties which will be used in the proof of our main results. Let $D = \{(t,s) : t \geq s \geq t_0\}$ and $D_0 = \{(t,s) : t > s > t_0\}$. We say that a function $H \in \mathbf{C}(D,\mathbb{R})$ belongs to the function class $\Im$, written as $H \in \Im$, if

(H1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on $D_0$;
(H2) $H$ has a continuous and nonpositive derivative in $D_0$ with respect to the second variable;
(H3) There exist functions $\rho \in \mathbf{C}^1([t_0,\infty), \mathbb{R}_+)$ and $h \in \mathbf{C}(D,\mathbb{R})$ such that

$$\frac{\partial}{\partial s}[H(t,s)\rho(s)] = -H(t,s)h(t,s), \quad (t,s) \in D_0.$$

Let $\rho \in \mathbf{C}([t_0,\infty), \mathbb{R}_+)$ and $H \in \Im$, we take an integral operator $A$ defined in [11], in terms of $H(t,s)$ and $\rho(s)$ as

$$A_T(\varphi; t) := \int_T^t H(t,s)\varphi(s)\rho(s)\mathrm{d}s, \quad t \geq T \geq t_0, \tag{1.4}$$

where $\varphi \in \mathbf{C}([t_0,\infty), \mathbb{R})$. It is easy seen that the integral operator $A$ satisfies the following properties:

$$A_T(l_1h_1 + l_2h_2; t) = l_1A_T(h_1;t) + l_2A_T(h_2;t); \tag{1.5}$$

$$A_T(h_3;t) \geq 0 \quad \text{whenever } h_3 \geq 0; \tag{1.6}$$

$$A_T(h_4';t) = -H(t,T)h_4(T)\rho(T) + A_T(\rho^{-1}h_4h;t). \tag{1.7}$$

Here, $h_1, h_2, h_3 \in \mathbf{C}([t_0,\infty), \mathbb{R})$, $h_4 \in \mathbf{C}^1([t_0,\infty), \mathbb{R})$, and $l_1, l_2 \in \mathbb{R}$.