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Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words

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ABSTRACT

In this paper, we apply an intuitionistic fuzzy quasi-metric version of a fixed point theorem, to obtain the existence of solution for a recurrence equation associated with the analysis of Quicksort algorithms.

1. Introduction and preliminaries

In this section, using the idea of intuitionistic fuzzy metric spaces introduced in [1–3] we define the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable (see [4]).

Lemma 1.1 ([5]). *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 1.2 ([6]). An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

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We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, a triangular norm $* = T$ on $[0, 1]$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \longrightarrow [0, 1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0, 1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \longrightarrow [0, 1]$ satisfying $S(0, x) = 0 \diamond x = x$, for all $x \in [0, 1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 1.3 (*[5]*). A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$, satisfying the following conditions:
$(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)
$(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$, (commutativity)
$(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$, (associativity)
$(\forall (x, x', y, y') \in (L^*)^4)\ (x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$. (monotonicity)

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then $\mathcal{T}$ is said to be a *continuous* t-norm.

Definition 1.4 (*[5]*). A continuous t-norm $\mathcal{T}$ on L^* is called continuous *t-representable* if and only if, there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now, define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \ldots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \ldots, x^{(n)}), x^{(n+1)})$$

for $n \geq 2$ and $x^{(i)} \in L^*$.

For example, $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* is a continuous t-representable.

Definition 1.5. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \longrightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined as, for all $x \in [0, 1]$, $N_s(x) = 1 - x$. We show $(N_s(\lambda), \lambda) = \mathcal{N}_s(\lambda)$.

Definition 1.6. The t-norm $\mathcal{T}$ is *Hadžić type* if for given $\varepsilon \in (0, 1)$ there is $\delta \in (0, 1)$, such that

$$\mathcal{T}^m(\mathcal{N}_s(\delta), \ldots, \mathcal{N}_s(\delta)) >_{L^*} \mathcal{N}_s(\varepsilon), \quad m \in \mathbf{N}.$$

A typical example of such t-norms is

$$\wedge(a, b) = (\min(a_1, b_1), \max(a_2, b_2)),$$

in which $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are two elements of L^*. The pair (X, d) is said to be a *quasi metric space*, if X is an arbitrary (non-empty) set, and d is a mapping from X^2 into $[0, +\infty)$, satisfying the following conditions for every $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 1.7. Let M, N are fuzzy sets from $X^2 \times [0, +\infty)$ to $[0, 1]$ such that $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy quasi-metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-representable and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times [0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 1.2) satisfying the following conditions for every $x, y, z \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x, y, 0) = 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t) = 1_{L^*}$ if and only if $x = y$;
(c) $\mathcal{M}_{M,N}(x, y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, s))$;
(d) $\mathcal{M}_{M,N}(x, y, \cdot) : [0, \infty) \longrightarrow L^*$ is left continuous.

In this case, $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy quasi-metric*. Here,

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)).$$

Note that an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfying for all $x, y \in X$ and $t > 0$ the symmetry axiom $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$ is an intuitionistic fuzzy metric space [4].

If the intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfies the condition:

$$\lim_{t \to \infty} \mathcal{M}_{M,N}(x, y, t) = \lim_{t \to \infty} \mathcal{M}_{M,N}(y, x, t) = 1_{L^*},$$

then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called *Menger* intuitionistic fuzzy quasi-metric space. If $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy quasi-metric space then $(X, \mathcal{M}^{-1}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy quasi-metric space, where $\mathcal{M}^{-1}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$. Moreover, if we denote by $\mathcal{M}^i_{M,N}$ the fuzzy set in $X^2 \times [0, +\infty)$ given by

$$\mathcal{M}^i_{M,N}(x, y, t) = \mathcal{T}(\mathcal{M}_{M,N}(x, y, t), \mathcal{M}^{-1}_{M,N}(x, y, t)),$$