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On the semilocal convergence of inexact Newton methods in Banach spaces

Ioannis K. Argyros *

Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA

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ABSTRACT

We provide two types of semilocal convergence theorems for approximating a solution of an equation in a Banach space setting using an inexact Newton method [I.K. Argyros, Relation between forcing sequences and inexact Newton iterates in Banach spaces, Computing 63 (2) (1999) 134–144; I.K. Argyros, A new convergence theorem for the inexact Newton method based on assumptions involving the second Fréchet-derivative, Comput. Appl. Math. 37 (7) (1999) 109–115; I.K. Argyros, Forcing sequences and inexact Newton iterates in Banach space, Appl. Math. Lett. 13 (1) (2000) 77–80; I.K. Argyros, Local convergence of inexact Newton-like iterative methods and applications, Comput. Math. Appl. 39 (2000) 69–75; I.K. Argyros, Computational Theory of Iterative Methods, in: C.K. Chui, L. Wuytack (Eds.), in: Studies in Computational Mathematics, vol. 15, Elsevier Publ. Co., New York, USA, 2007; X. Guo, On semilocal convergence of inexact Newton methods, J. Comput. Math. 25 (2) (2007) 231–242]. By using more precise majorizing sequences than before [X. Guo, On semilocal convergence of inexact Newton methods, J. Comput. Math. 25 (2) (2007) 231–242; Z.D. Huang, On the convergence of inexact Newton method, J. Zheijiang University, Nat. Sci. Ed. 30 (4) (2003) 393–396; L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982; X.H. Wang, Convergence on the iteration of Halley family in weak condition, Chinese Sci. Bull. 42 (7) (1997) 552–555; T.J. Ypma, Local convergence of inexact Newton methods, SIAM J. Numer. Anal. 21 (3) (1984) 583–590], we provide (under the same computational cost) under the same or weaker hypotheses: finer error bounds on the distances involved; an at least as precise information on the location of the solution. Moreover if the splitting method is used, we show that a smaller number of inner/outer iterations can be obtained.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a real Banach space X with values in a real Banach space Y.

A large number of problems in applied mathematics and also in engineering is solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ for some suitable operator Q, where x is the state. Then, the equilibrium states are determined

* Tel.: +1 580 5368754; fax: +1 580 5812616.
E-mail address: iargyros@cameron.edu.

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by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We shall use the iterative procedure

$$x_{n+1} = x_n + s_n, \quad (n \geq 0), \tag{1.2}$$

where step s_n satisfies

$$F'(x_n)s_n = -F(x_n) + r_n \quad (n \geq 0), \tag{1.3}$$

for some null residual sequence $\{r_n\} \subseteq Y$, to generate a sequence $\{x_n\}$ approximating the solution x^*.

A convergence analysis of inexact Newton method (1.2) has been given by many authors and under various assumptions [1–4,7–10,12,14–17]. If $s_n = 0$ $(n \geq 0)$, we obtain the ordinary Newton's method for solving nonlinear equations. Otherwise iterative procedure (1.2) is called inexact Newton's method. By semilocal convergence we mean that we are seeking a solution x^* inside a ball centered at the initial guess x_0, and of a certain finite radius. We recommend the reading of Chapter XVIII on Newton's method of the Kantorovich and Akilov book [15], especially Theorem 6 in Section 1.5, along with the proof, to see how the majorizing function is constructed there (whose least zero plays an important role) (see also relevant Section 4.2 in [7]).

There are two kinds of methods for the solution of linear equations. The first kind of methods are the so-called direct methods, or elimination methods. In this case the exact solution is determined through a finite number of arithmetic operations in real arithmetic without considering the round-off errors. For a list of difficulties and how to handle them we refer the reader to [9].

Another kind of methods are the iterative ones, which result in a two-stage method, or sometimes termed as inner/outer iterations for solving nonlinear equation (1.1).

In this study we are motivated by optimization considerations and the elegant works in [12,14,16]. Guo provided semilocal convergence analysis for inexact Newton method (1.2) using Lipschitz conditions on the Fréchet-derivative F' of operator F. He also provided bounds on the number of inner iteration steps.

We use a combination of Lipschitz and center-Lipschitz conditions along the lines of our works on Newton as well as Newton-like methods [5–7] to provide a new convergence analysis for inexact Newton method (1.2) with advantages over earlier works [1–4,7–10,12,14–17] (especially [12,14–17]) as stated in the abstract of this paper.

2. Type I semilocal convergence analysis of inexact Newton method (1.2)

The main new idea is to introduce a center-Lipschitz condition (with constant γ_0), and then use it instead of the Lipschitz condition (with constant γ) employed in [12] to provide more precise upper bounds on the norms $\|F'(x)^{-1}\, F'(x_0)\|$ in case $\gamma_0 < \gamma$ (see also the proof of Theorem 2.1, and Remark 2.2 that follow). We can show the main semilocal convergence result for the inexact Newton method (1.2):

Theorem 2.1. *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose: $F'(x_0)^{-1} \in L(Y, X)$ for some $x_0 \in D$, and there exist parameters $\beta > 0$, $\gamma_0 \geq 0$, $\gamma \geq 0$, and $\eta \in [0, 1)$ such that for all $x, y \in D$:*

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_0)\| \leq \beta, \\
&\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \gamma_0\|x - x_0\|, \\
&\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \gamma\|x - y\|, \\
&\frac{\|F'(x_0)^{-1}r_n\|}{\|F'(x_0)^{-1}F(x_n)\|} \leq \eta_n, \quad \eta = \max_n\{\eta_n\}, \\
&\beta\gamma \leq p_0,
\end{aligned}$$

and

$$U_1 = \overline{U}\left(x_0, \frac{s_1}{1-\sigma}\right) = \left\{x \in X: \|x - x_0\| \leq \frac{s_1}{1-\sigma}\right\} \subseteq D,$$

where,

$$p_0 = -\frac{2\eta^2 + 14\eta + 11 - \sqrt{(4\eta+5)^3}}{(1+\eta)(1-\eta)^2},$$