Journal of Computational and Applied Mathematics 228 (2009) 444–457

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Eigenvalue bounds for the Schur complement with a pressure convection–diffusion preconditioner in incompressible flow computations

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ARTICLE INFO

Article history:
Received 5 September 2007
Received in revised form 28 May 2008

MSC:
65F10
65F35
65F22
65N25

Keywords:
Non-symmetric saddle point problems
Preconditioning
Schur complement
Finite element methods
Navier–Stokes equations

ABSTRACT

If the stationary Navier–Stokes system or an implicit time discretization of the evolutionary Navier–Stokes system is linearized by a Picard iteration and discretized in space by a mixed finite element method, there arises a saddle point system which may be solved by a Krylov subspace method or an Uzawa type approach. For each of these resolution methods, it is necessary to precondition the Schur complement associated to the saddle point problem in question. In the work at hand, we give upper and lower bounds of the eigenvalues of this Schur complement under the assumption that it is preconditioned by a pressure convection–diffusion matrix.

1. Introduction

Consider the time-dependent Navier–Stokes equations,

$$\partial_t u - \nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \operatorname{div} u = 0, \tag{1.1}$$

or its stationary counterpart,

$$-\nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \operatorname{div} u = 0, \tag{1.2}$$

supplemented by boundary conditions and, in the time-dependent case, by initial conditions. Suppose this (initial-) boundary value problem is discretized implicitly or semi-implicitly in time (if there is a time variable), and is linearized by a Picard iteration (if the problem is stationary or was implicitly discretized in time). Further suppose it is discretized in space by a mixed finite element method. In this situation, a variational problem of the following type arises: Find $u_h \in V_h$, $\pi_h \in P_h$ such that

$$a(u_h, w) + b_1(w, \pi_h) = \mathfrak{F}(w) \quad \text{for } w \in V_h, \qquad b_2(u_h, \sigma) - c(\pi_h, \sigma) = \mathfrak{G}(\sigma) \quad \text{for } \sigma \in P_h. \tag{1.3}$$

Here h is a grid parameter, V_h and P_h are finite dimensional spaces, $\mathfrak{F} : V_h \mapsto \mathbb{R}$ and $\mathfrak{G} : P_h \mapsto \mathbb{R}$ are linear operators, b_1 and b_2 are bilinear forms corresponding to respectively the gradient and the divergence operator, and a is a bilinear form representing an "advection–diffusion–reaction operator" of the form $-\nu \cdot \Delta u + (v_0 \cdot \nabla)u + \theta \cdot u$. The parameter θ corresponds

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doi:10.1016/j.cam.2008.10.017

to the inverse of the time step in the evolutionary case, and equals 0 otherwise. The function v_0 is the velocity approximation from the preceding step of the nonlinear iteration or from the preceding time step. In the case of LBB-stable mixed finite element methods, the form c vanishes; otherwise it plays the role of a "stabilization term". Other such terms may appear in the definition of b_1, b_2 and a, or may be incorporated into $\mathfrak{F}$ and $\mathfrak{G}$. In the LBB case, the forms b_1 and b_2 usually coincide. Algebraically, problem (1.3) corresponds to a saddle point system of the form

$$K \cdot \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \quad \text{with } K := \begin{pmatrix} N & B_1^{\mathrm{T}} \\ B_2 & -C \end{pmatrix}, \tag{1.4}$$

where N may be considered as a "vector advection–diffusion–reaction operator". The matrices B_1, B_2 are discrete gradient and divergence operators, possibly including stabilization terms; C is a stabilization matrix which is zero in the case of LBB-stable finite elements. The solution vector X corresponds to the unknown u_h in (1.3), and the vector P to π_h. Since usually the size of K is large, iterative methods frequently are the most efficient means for solving (1.4). Following [20], we may distinguish two major classes of such solvers, that is, multigrid methods and Krylov subspace methods like GMRES. In general, the latter methods are used in two different ways: either they are applied to the global matrix K, or first the velocity part X is eliminated, and then the pressure part P is computed by solving a system with the pressure Schur complement $S := C + B_2 \cdot N^{-1} \cdot B_1^{\mathrm{T}}$ as system matrix. As explained in [20], in both cases (not only in the second), and also in the case of some multigrid methods, a crucial problem consists in finding a suitable preconditioner for S. Under the assumption that the discrete advection–diffusion–reaction operator N can be efficiently approximated, such a preconditioner was proposed in [15]; it will be denoted by $\hat{S}^{-1}$ in what follows, and is given by $\hat{S}^{-1} := M_p^{-1} \cdot N_p \cdot A_p^{-1}$, where M_p and A_p are projections of the identity and of a Neumann Laplacian onto the pressure finite element space, and N_p is the projection of the velocity operator $-\nu \cdot \Delta u + (v_0 \cdot \nabla)u + \theta \cdot u$ onto the same space.

This choice of preconditioner is motivated in [15,19] and [9, p. 347–348] for example. As concerns numerical tests, a great number of them have been performed by now, with very satisfactory results. We refer to [6–9,15,19–21,24,26] in this respect. As concerns other aspects of solving (1.4), like symmetric preconditioners, multigrid methods, or the case of exterior flows, we mention [3,4,16–18,23,25,28]. This list is by no means exhaustive; many more references may be found in [9].

In the work at hand, we are interested in a theoretical aspect: we want to determine upper and lower bounds of the eigenvalues of $\hat{S}^{-1} \cdot S$. These bounds are crucial in attempts to evaluate the performance of iterative methods applied to (1.4); compare [9, Chapter 4]. Partial results on such bounds were presented in [7] (Newton's method) and [8]; a detailed theory was given in [19]. In the latter article, it was shown in particular how to treat a large class of stabilized methods in a unified way. The arguments in [19] are largely based on matrix algebra, but they also refer to H^2-estimates of solutions to elliptic partial differential equations. These estimates, besides requiring unnecessary restrictions on the domain of solutions to (1.1) and (1.2), present the additional inconvenience that the constants appearing in them are not very explicit as concerns their dependence on the parameters of the problem at hand. But it is precisely this dependence which is of interest in view of performance analysis of iterative methods.

In the present paper we will present a theory which is self-contained, does not use any regularity results for partial differential equations, and allows us to trace all relevant parameters in an explicit way. Our arguments are based on a variational approach which we already used in [5] in order to deal with preconditioning of the Schur complement by a pressure mass matrix. In the present context, this approach consists in writing the eigenvalue equation $\hat{S}^{-1} \cdot S \cdot P = \lambda \cdot P$ as a variational problem, estimating the solutions of this problem, and then deducing from these estimates the desired bounds of λ. This program will be developed in the form of an abstract theory (Section 2), which is afterwards applied to the stabilized finite element methods considered in [19], and to LBB-stable methods (Section 3).

2. Abstract theory

Let V and M be finite dimensional Hilbert spaces with scalar products denoted by respectively $(\,,\,)_V$ and $(\,,\,)_M$, and with associated norms denoted by $\|\ \|_V$ and $\|\ \|_M$. Since we want to deal with the two cases of enclosed and non-enclosed flow at the same time, we fix some $m_0 \in M$. Typically the case $m_0 = 0$ is related to models of non-enclosed flow, whereas the case $m_0 \neq 0$ pertains to enclosed flows. We put $M_0 := \{p \in M : (p, m_0)_M = 0\}$. Of course, if $m_0 = 0$, we have $M = M_0$.

Moreover, we introduce another norm on V, denoted by $\|\ \|_a$ and supposed to be induced by a scalar product. This assumption and the fact that the dimension of V is finite ensure that $\|\ \|_a$ is a norm induced by the scalar product of a Hilbert space. But the scalar product in question will not appear explicitly. The norms $\|\ \|_V$ and $\|\ \|_a$ are assumed to be linked by the inequality

$$\|v\|_V \le K_1 \cdot \|v\|_a \quad \text{for } v \in V, \tag{2.1}$$

with some constant $K_1 > 0$. Next consider bilinear forms $a : V \times V \mapsto \mathbb{R}$, $b_1, b_2 : V \times M \mapsto \mathbb{R}$, $c : M \times M \mapsto \mathbb{R}$ such that c is symmetric and $c(p, p) \ge 0$ for $p \in M$, and such that there are constants $\epsilon \in [0, \infty)$, $K_2, \ldots, K_5 \in (0, \infty)$ with

$$K_2 \cdot \|v\|_a^2 \le a(v, v), \qquad |a(v, w)| \le K_3 \cdot \|v\|_a \cdot \|w\|_V \quad \text{for } v, w \in V; \tag{2.2}$$

$$|b_1(v, p)| \le K_4 \cdot \|v\|_V \cdot \|p\|_M, \quad |b_1(v, p) - b_2(v, p)| \le \epsilon \cdot \|v\|_a \cdot \|p\|_M \quad \text{for } v \in V, p \in M; \tag{2.3}$$

$$|c(p, q)| \le K_5 \cdot \|p\|_M \cdot \|q\|_M \quad \text{for } p, q \in M; \tag{2.4}$$

$$b_2(v, m_0) = 0 \quad \text{for } v \in V, \qquad c(m_0, p) = 0 \quad \text{for } p \in M. \tag{2.5}$$