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# An evaluation of Clenshaw–Curtis quadrature rule for integration w.r.t. singular measures

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ABSTRACT

This work is devoted to the study of quadrature rules for integration with respect to (w.r.t.) general probability measures with known moments. Automatic calculation of the Clenshaw–Curtis rules is considered and analyzed. It is shown that it is possible to construct these rules in a stable manner for quadrature w.r.t. balanced measures. In order to make a comparison Gauss rules and their stable implementation for integration w.r.t. balanced measures are recalled. Convergence rates are tested in the case of binomial measures.



## 1. Introduction

In quadrature theory, much effort has been done in the analysis of the integration with respect to (w.r.t.) the Lebesgue measure or to some of its weighted variants. Among the possible generalizations of the problem, the case of singular measures naturally appears, for instance, when dealing with fractal properties of some physical phenomenon, see [3,17].

In a recent review paper [25], Trefethen compares the convergence rates of Clenshaw–Curtis rules with the Gauss ones. In this paper the author points out that the two rates of convergence are similar if the integrand function is not analytic in a suitable neighborhood of the interval of integration. In the present paper we want to compare the same two families of quadrature rules when the integration is performed w.r.t. a singular (fractal) measure.

We begin with the introduction of the convergence theory for general quadrature rules in Section 2. Then in Section 3, we introduce the Clenshaw–Curtis and Gauss families of quadrature rules and their numerical construction. On the one hand we notice that these rules converge for wide classes of functions. On the other hand, for a general measure, we observe that the automatic calculation passes through an unstable procedure which is of different origin in the two cases. In the Gauss quadrature it appears when the construction of the recurrence coefficients for orthogonal polynomials is carried out [4], while in the Clenshaw–Curtis case when the calculation of modified moments is performed [5]. In Section 4 we recall the definition of balanced measures. We show that, despite the general case, for this class of singular measures it is possible to construct in a stable manner both formulae. In the case of Gauss quadrature this has been developed in [15], while for Clenshaw–Curtis rule it is made adapting the analysis in [24]. In the same section the connection with the theory of linear refinable functionals introduced in [14] is also analyzed. As an application, in Section 5 the quadrature w.r.t. binomial measures is performed through numerical tests.

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## 2. Preliminary results and definitions

In this introduction we present some results valid for a general measure $\mu$, that we will assume finite, positive and defined in a closed interval $[a, b]$. Our aim is to study how to calculate

$$I_\mu(f) \equiv \int_a^b f(x)\mathrm{d}\mu(x); \tag{1}$$

where $f \in L^1_\mu \equiv \{f : [a, b] \to \mathbb{R} : \int_a^b |f(x)|\mathrm{d}\mu(x) < \infty\}$.

In general a quadrature rule $\mathbb{I}_n$ is defined by means of $(n + 1)$ distinct points $\xi_j \in [a, b]$ called nodes and $(n + 1)$ real values $w_j$ called weights:

$$\mathbb{I}_n(f) \equiv \sum_{j=0}^{n} w_j f(\xi_j). \tag{2}$$

In order to obtain efficient quadrature rules, we can construct $\mathbb{I}_n$ to be the exact integral of an approximating function $\tilde{f}$: $\mathbb{I}_n(f) = I_\mu(\tilde{f})$. In what follows, we assume that the moments of the measure are known:

$$\lambda_j \equiv \int_a^b x^j \mathrm{d}\mu(x) \quad \forall j = 0, 1, \ldots \tag{3}$$

and for this reason we will take as approximating function a polynomial, $\tilde{f}(x) \in \mathbb{P}^n$ where $\mathbb{P}^n$ are the polynomials of degree at most $n$. Such rule will be called interpolatory quadrature formula when the polynomial that we integrate exactly is the (unique) polynomial of degree $n$ interpolating the function $f$ at the nodes $\xi_j$.

We will say that a quadrature rule has degree of exactness $d$ if

$$\sum_{j=0}^{n} w_j \xi_j^q = \lambda_q \quad \forall q \leq d,\ q \in \mathbb{N}.$$

It is well known that every quadrature rule with $n + 1$ nodes of degree of exactness at least $n$ is interpolatory. In general the following result holds true, see [6, Section 1.3]:

**Theorem 2.1.** *The quadrature rule (2) has degree of exactness $d = n + k,\ k \geq 0$ if and only if both of the following conditions are satisfied:*

1. *the formula (2) is interpolatory;*
2. *the following holds true:*

$$\int_a^b \omega_n(x)p(x)\mathrm{d}\mu(x) = 0 \quad \forall p \in \mathbb{P}^{k-1}\ (\mathbb{P}^{-1} \equiv \emptyset)$$

*where $\omega_n(x) = \prod_{j=0}^{n}(x - \xi_j)$ is the nodal polynomial.*

Given a function $f \in L^1_\mu$, we will say that a sequence of quadrature rules $\{\mathbb{I}_n\}_n$ converges in $f$ if $\mathbb{I}_n(f) \to_n I_\mu(f)$.

Given a function $f \in C^0$, we will denote by $p_d^*(x)$ the polynomial[1] of degree at most $d$ that gives the best approximation to $f$ on $[a, b]$ w.r.t. the supremum norm. We will also denote by $E_d^* \equiv \|f - p_d^*\|_\infty$. With this notations, the following theorem gives the most general error estimate, see [12, Theorem 5.2.2] or [25, Theorem 4.1].

**Theorem 2.2.** *Let $\mathbb{I}_n$ be a quadrature rule with weights $w_j, j = 0, \ldots, n$ of degree of exactness $d \geq 0$. Then for all $f \in C^0$ we have:*

$$\left|I_\mu(f) - n(f)\right| \leq E_d^* \left[\sum_{j=0}^{n} |w_j| + \mu([a, b])\right].$$

The result is proved simply applying the definitions and the triangular inequality.

If we consider a family of rules $\{\mathbb{I}_n\}_n$ of increasing degrees of exactness $d_n$ and such that $\sum_{j=0}^{n} |w_j| \leq K_n$ we will have that the rule converges if $K_n E_{d_n}^* \to_{n\to\infty} 0$. Notice that for interpolatory quadrature rules the constant $K_n$ is bounded from above by the Lebesgue constant $\Lambda_n$ (see [22, Eq. (8.11)]).

As corollary of the Weierstrass theorem we can state also that for every $f \in C^0$ there exists always a sequence of polynomials uniformly convergent to $f$, and therefore the corresponding quadrature rules will be convergent. On the other hand it is very well known that equispaced interpolatory quadrature formulae do not converge in general due to Runge phenomenon.

[1] Note that this polynomial is unique. For the theory of the best approximation see, ad example, [19, Section 3.2].