Journal of Computational and Applied Mathematics 231 (2009) 541–551

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Iterative methods of order four and five for systems of nonlinear equations☆

Alicia Cordero*, Eulalia Martínez, Juan R. Torregrosa

Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Camino de Vera, s/n, 46022, Valencia, Spain
Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera, s/n, 46022, Valencia, Spain

ARTICLE INFO

Article history:
Received 24 July 2008
Received in revised form 27 March 2009

Keywords:
Nonlinear systems
Adomian decomposition
The Newton method
Fixed point iteration
Convergence order

ABSTRACT

The Adomian decomposition is used in order to obtain a family of methods to solve systems of nonlinear equations. The order of convergence of these methods is proved to be $p \geq 2$, under the same conditions as the classical Newton method. Also, numerical examples will confirm the theoretical results.

1. Introduction

The main goal of this work is to obtain new iterative formulas in order to solve systems of nonlinear equations. They are proved to be modifications on the classical Newton method which accelerate the convergence of the iterative process.

In previous works, the authors have obtained variants on the Newton method based on quadrature formulas whose truncation error was up to $O(h^5)$ (see [1,2]). Indeed, a general interpolatory quadrature formula is used in [3] in order to obtain a family of modified Newton methods with order of convergence up to $2d + 1$, when the partial derivatives of each coordinate function in the solution, from order two until d, are zero. Moreover, in [3], we modify the general method from [4] getting a collection of multipoint iterative methods obtained from the Newton method by replacing $F(x^{(k)})$ by a linear combination of values of $F(x)$ in different points.

Nevertheless, the approach used in this paper to solve a nonlinear system is different: by using Adomian polynomials, we obtain a family of multipoint iterative formulas, which include the Newton and Traub (see [5]) methods in the simplest cases.

The decomposition method using Adomian polynomials is used to solve different problems on applied mathematics in [6]. Indeed, Babolian et al. (see [7]) apply this general method to a concrete nonlinear system. Nevertheless, with a different system, it is necessary to repeat all the process. In [8], Adomian decomposition method is applied to construct some numerical algorithms for solving systems of two nonlinear equations.

We deduce in Section 2, by means of Adomian decomposition, a family of iterative formulas that can be applied to solve any nonlinear system without knowledge about Adomian polynomials. These iterative formulas involve classical methods, like those of Newton (order $p = 2$) and Traub (order $p = 3$), and also new methods whose convergence order is proved to be higher.

☆ This research was supported by Ministerio de Ciencia y Tecnología MTM2007-64477.
* Corresponding author at: Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera, s/n, 46022, Valencia, Spain.
E-mail addresses: acordero@mat.upv.es (A. Cordero), eumarti@mat.upv.es (E. Martínez), jrtorre@mat.upv.es (J.R. Torregrosa).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.04.015

Now, we remember the most common notions about nonlinear systems and the convergence of an iterative method.

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution $\bar{x} \in D$ of the nonlinear system $F(x) = 0$, of n equations with n variables. This solution can be obtained as a fixed point of some function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of the fixed point iteration method

$$x^{(k+1)} = G(x^{(k)}), \quad k = 0, 1, \ldots, \tag{1}$$

where $x^{(0)}$ is the initial estimation. The best known fixed point method is the classical Newton method, given by

$$x^{(k+1)} = x^{(k)} - J_F(x^{(k)})^{-1} F(x^{(k)}), \quad k = 0, 1, \ldots, \tag{2}$$

where $J_F(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration $x^{(k)}$.

Definition 1. Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ convergent to $\bar{x}$. Then, convergence is called

(a) *linear*, if there exist M, $0 < M < 1$, and k_0 such that

$$\left\| x^{(k+1)} - \bar{x} \right\| \leq M \left\| x^{(k)} - \bar{x} \right\|, \quad \forall k \geq k_0. \tag{3}$$

(b) *of order* p, $p \geq 2$, if there exist M, $M > 0$, and k_0 such that

$$\left\| x^{(k+1)} - \bar{x} \right\| \leq M \left\| x^{(k)} - \bar{x} \right\|^p \quad \forall k \geq k_0. \tag{4}$$

Definition 2 (*See [9]*). Let $\bar{x}$ be a zero of the function F and suppose that $x^{(k-1)}$, $x^{(k)}$ and $x^{(k+1)}$ are three consecutive iterations close to $\bar{x}$. Then, the computational order of convergence ρ can be approximated using the formula

$$\rho \approx \frac{\ln\left(\left\| x^{(k+1)} - \bar{x} \right\| / \left\| x^{(k)} - \bar{x} \right\|\right)}{\ln\left(\left\| x^{(k)} - \bar{x} \right\| / \left\| x^{(k-1)} - \bar{x} \right\|\right)}. \tag{5}$$

In addition, in order to compare different methods, we use the efficiency index, $p^{1/d}$ (see [10]), where p is the order of convergence and d is the total number of new functional evaluations (per iteration) required by the method.

In Section 3, we study the convergence of the different methods by using the following result.

Theorem 1 (*See [5]*). *Let $G(x)$ be a fixed point function with continuous partial derivatives of order p with respect to all components of x. The iterative method $x^{(k+1)} = G(x^{(k)})$ is of order p if*

$$G(\bar{x}) = \bar{x};$$
$$\frac{\partial^k g_i(\bar{x})}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}} = 0, \quad \text{for all } 1 \leq k \leq p-1,\ 1 \leq i, j_1, \ldots, j_k \leq n;$$
$$\frac{\partial^p g_i(\bar{x})}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_p}} \neq 0, \quad \text{for at least one value of } i, j_1, \ldots, j_p$$

where g_i are the component functions of G.

Finally, numerical tests are made in Section 4 comparing the classical and new methods, confirming the theoretical results.

2. Description of the methods

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n > 1$, be a sufficiently differentiable function whose coordinate functions are $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots, n$. Let $\bar{x}$ be a zero of the nonlinear system $F(x) = 0$ and $\alpha \in \mathbb{R}^n$ an estimation of $\bar{x}$. Then, this system is equivalent to:

$$F(\alpha) + J_F(\alpha)(x - \alpha)^{\mathrm{T}} + K(x) = 0,$$

where $J_F(\alpha)$ is the Jacobian matrix of the function F evaluated in the estimation α and $K : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ verifies:

$$K(x) = F(x) - F(\alpha) - J_F(\alpha)(x - \alpha)^{\mathrm{T}}.$$

Then,

$$x = \alpha - J_F^{-1}(\alpha)F(\alpha) - J_F^{-1}(\alpha)K(x).$$

Let us denote the linear component as $c \equiv \alpha - J_F^{-1}(\alpha)F(\alpha) \in \mathbb{R}^n$, and by $P(x)$ the nonlinear one, $P(x) = -J_F^{-1}(\alpha)K(x)$, $P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with coordinate functions $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. So,

$$x = c + P(x).$$