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Semi-analytic integration of hypersingular Galerkin BIEs for three-dimensional potential problems☆

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ABSTRACT

An accurate and efficient semi-analytic integration technique is developed for three-dimensional hypersingular boundary integral equations of potential theory. Investigated in the context of a Galerkin approach, surface integrals are defined as limits to the boundary and linear surface elements are employed to approximate the geometry and field variables on the boundary. In the inner integration procedure, all singular and non-singular integrals over a triangular boundary element are expressed exactly as analytic formulae over the edges of the integration triangle. In the outer integration scheme, closed-form expressions are obtained for the coincident case, wherein the divergent terms are identified explicitly and are shown to cancel with corresponding terms from the edge-adjacent case. The remaining surface integrals, containing only weak singularities, are carried out successfully by use of standard numerical cubatures. Sample problems are included to illustrate the performance and validity of the proposed algorithm.

1. Introduction

The computational treatment of three-dimensional elliptic boundary-value problems via the boundary integral equation (BIE) method often involves hypersingular BIEs [1,2]. In the boundary element method (BEM), hypersingular BIEs are essential when dealing with crack problems [3], symmetric Galerkin BIEs [4] and error estimation methods [5]. It is therefore important to design effective and reliable techniques for evaluating accurately and efficiently hypersingular surface integrals.

To this end, several approaches have been proposed in the literature to deal with the double surface integrals featuring strongly-singular and hypersingular kernels associated with a Galerkin approximation. For three-dimensional problems, regularization techniques based on Stokes theorem that require reformulation of the integrands to reduce the order of singularity to weakly-singular BIEs can be found in [6,7]. The resulting weakly-singular integrals are carried out numerically using ordinary methods. In addition, one can mention direct evaluation schemes [8–10] based on Hadamard finite part definition to assign finite values to potentially unbounded integrals. In these Hadamard finite part strategies, the inner integrals are calculated exactly and the outer integrals are computed with suitable numerical cubatures.

In another direct evaluation method called *limit to the boundary approach* [11,12], the divergent terms in hypersingular integrals are explicitly identified and removed leaving integrals that can be handled via standard cubatures. In the limit to

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the boundary technique, there is no reformulation of the underlying integral equations, and there is no need for Cauchy principal value and Hadamard finite part definitions.

In view of the inherent challenges associated with a comprehensive and efficient treatment of element integrals stemming from a Galerkin discretization of hypersingular BIEs over polygonal boundaries, and in order to design an accurate and stable general purpose algorithm for solving three-dimensional problems of potential theory, a semi-analytic technique that is based on the limit to the boundary approach is proposed in this article. Upon utilizing piecewise linear shape functions over flat triangles, the inner integration procedure is carried out exactly by means of the recursive formulae developed in [13]. In the outer integration scheme, closed-form expressions are derived for the coincident case, wherein the divergent terms are explicitly identified. Due to the edge-by-edge feature of the foregoing methodology, analytic expressions together with divergent terms have also been obtained for the common edge in the edge-adjacent case. These divergent terms in the edge-adjacent case are shown to cancel with corresponding terms from the coincident integration. With the aid of several Duffy transformations [14,15], the weak singularities in the remaining edge-adjacent case and vertex-adjacent integration are completely removed leaving only regular integrals that are handled by well-established cubature methods to any desired accuracy.

It should be noted that the foregoing technique has also been applied to the singular Galerkin BIEs of potential theory involving only weakly-singular and strongly-singular kernels. Exact formulae have also been obtained for the coincident case and for the common edge in the edge-adjacent case.

Details of the foregoing limit to the boundary approach, including the explicit identification of divergent terms in hypersingular Galerkin integrals, are elucidated. Numerical examples are included to illustrate the performance and validity of the proposed method.

2. Problem formulation

Consider the numerical treatment of the Laplace equation $\nabla^2 u = 0$ in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ via a BIE method. To solve the foregoing problem, let $\overline{\Omega} = \Omega \cup \Gamma$, and for some $\mathbb{R} \ni \varepsilon > 0$, select $\boldsymbol{x}_\varepsilon \in \mathbb{R}^3 \setminus \overline{\Omega}$ to be an exterior point to $\overline{\Omega}$. In addition, let $\boldsymbol{n}$ be the unit normal to Γ directed towards the exterior of Ω and denote by $t = \partial u/\partial n$ the flux associated with the potential u.

On employing the integral representation formula for the gradient of u [1], the potential u and flux t on Γ can be obtained by solving the hypersingular BIE

$$\boldsymbol{n}(\boldsymbol{x}) \cdot \left\{ \lim_{\boldsymbol{x}_\varepsilon \to \boldsymbol{x} \in \Gamma} \left(\int_\Gamma \boldsymbol{H}(\boldsymbol{x}_\varepsilon, \boldsymbol{y})\, t(\boldsymbol{y})\, \mathrm{d}\Gamma_{\boldsymbol{y}} + \int_\Gamma \boldsymbol{T}(\boldsymbol{x}_\varepsilon, \boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}\Gamma_{\boldsymbol{y}} \right) \right\} = 0, \tag{1}$$

where the strongly-singular and hypersingular kernels, $\boldsymbol{H}$ and $\boldsymbol{T}$, are given respectively by

$$\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi} \frac{\boldsymbol{x} - \boldsymbol{y}}{\|\boldsymbol{x} - \boldsymbol{y}\|^3}, \quad \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi} \left[\frac{\mathbf{I}_2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3} - 3 \frac{(\boldsymbol{x} - \boldsymbol{y}) \otimes (\boldsymbol{x} - \boldsymbol{y})}{\|\boldsymbol{x} - \boldsymbol{y}\|^5} \right], \quad \boldsymbol{x} \neq \boldsymbol{y}. \tag{2}$$

In (2), $\mathbf{I}_2$ is the symmetric, second-order identity tensor; the variable $\boldsymbol{x} \in \mathbb{R}^3$ is frequently called the source point whereas $\boldsymbol{y} \in \mathbb{R}^3$ is termed the receiver or field point. Note that the numerical treatment presented in this study is applicable regardless of the types of boundary conditions, i.e., Dirichlet, Neumann or mixed boundary conditions.

2.1. Galerkin approximation

With reference to Fig. 1, assume that Γ is the surface a polyhedron and consider a triangulation of $\Gamma = \bigcup \overline{E_q}$ into closed and non-overlapping surface elements such that E_q is an open flat triangle. Let N_E be the total number of elements (triangles) on Γ. A usual procedure in the numerical treatment of (1) is to expand the boundary potential u and flux t in terms of respective nodal values and basis shape functions ψ_j at discrete points $\boldsymbol{y}^j$ on Γ as follows

$$u(\boldsymbol{y}) = \sum_{j=1}^{N} u(\boldsymbol{y}^j)\, \psi_j(\boldsymbol{y}), \qquad t(\boldsymbol{y}) = \sum_{j=1}^{N} t(\boldsymbol{y}^j)\, \psi_j(\boldsymbol{y}), \quad \boldsymbol{y}^j, \boldsymbol{y} \in \Gamma, \tag{3}$$

where N is the total number of boundary nodes on Γ.

With such definitions, a Galerkin approach for solving (1) requires that

$$\int_\Gamma \psi_i(\boldsymbol{x})\, \boldsymbol{n}(\boldsymbol{x}) \cdot \left\{ \lim_{\boldsymbol{x}_\varepsilon \to \boldsymbol{x} \in \Gamma} \left(\int_\Gamma \boldsymbol{H}(\boldsymbol{x}_\varepsilon, \boldsymbol{y})\, t(\boldsymbol{y})\, \mathrm{d}\Gamma_{\boldsymbol{y}} + \int_\Gamma \boldsymbol{T}(\boldsymbol{x}_\varepsilon, \boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}\Gamma_{\boldsymbol{y}} \right) \right\} \mathrm{d}\Gamma_{\boldsymbol{x}} = 0. \tag{4}$$

With the aid of (3), Eq. (4) reduces to a dense linear system for boundary potential u and flux t as

$$\mathbf{G}\{t\} = \mathbf{H}\{u\}, \tag{5}$$