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On a graded mesh method for a class of weakly singular Volterra integral equations

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ABSTRACT

In this paper a class of weakly singular Volterra integral equations with an infinite set of solutions is investigated. Among the set of solutions only one particular solution is smooth and all others are singular at the origin. The numerical solution of this class of equations has been a difficult topic to analyze and has received much previous investigation. The aim of this paper is to improve the convergence rates by a graded mesh method. The convergence rates are proved and a variety of numerical examples are provided to support the theoretical findings.

1. Introduction

In this paper we solve the following weakly singular equation

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s)\mathrm{d}s + g(t), \quad t \in (0, T], \tag{1}$$

where $\mu > 0$ and g is a given function. This class of equations arises in a certain type of heat conduction problems with time dependent boundary conditions (see e.g., in [1]). The numerical solutions of the equations have been investigated by several authors (see e.g., in [2,1,3–6]). The following analytical results of the solutions to (1) are given in [7].

(a) If $0 < \mu \le 1$ and $g \in C^1[0, T]$ (with $g(0) = 0$ for $\mu = 1$), Eq. (1) has an infinite set of continuous solutions which are given by the formula

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))\mathrm{d}s, \tag{2}$$

where

$$\gamma = \begin{cases} \dfrac{g(0)}{\mu - 1} & \text{if } \mu < 1, \\ 0 & \text{if } \mu = 1, \end{cases}$$

and c_0 is an arbitrary constant. The set of solutions contain only one particular solution which belongs to $C^1[0, T]$ (corresponding to $c_0 = 0$).

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(b) If $\mu > 1$ and $g \in C^m[0,T]$ ($m \geq 0$), the unique solution $u \in C^m[0,T]$ is

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s) \mathrm{d}s.$$

Lima and Diogo [1,3] analyze the Euler method for solving Eq. (1) and obtain a convergence rate $O(h^\mu)$ for $0 < \mu \leq 1$, where h stands for the maximum mesh size. Later Diogo et al. [4] used a coordinate transformation $t = \tau + \epsilon$ where ϵ is either a constant or a value depending on the mesh size h. They obtained a convergence rate like $O(h/\epsilon)$ which definitely improves the previous results. However due to the fact that ϵ is not allowed to be too large, the convergence rate is still not optimal. In this paper we try to apply graded meshes, which has been developed by Brunner [8](see also in [9]) for solving integral equation with weakly singular convolutionary kernels, to the equation. We obtain for the Euler method on graded meshes with the grading exponent $r \geq 2$ that $O(N^{-1})$ for $\mu \geq 1$, $O(N^{-1} \ln N)$ for $1/r \leq \mu < 1$ and $O(N^{-r\mu})$ for $0 < \mu < 1/r$.

Throughout this paper, we use the letters C and c (or with subscripts) to denote generic positive constants which are independent of the discretizing process.

2. Graded mesh method

For a positive integer N, define a mesh

$$\Pi_N = \{t_0, \ldots, t_N : 0 = t_0 < t_1 < \cdots < t_N = T\},$$

by

$$t_j = T\left(\frac{j}{N}\right)^r, \quad j = 0, \ldots, N, \tag{3}$$

where the real number $r \geq 2$, which is called the grading exponent, characterizes the nonuniformity of the mesh. The mesh points are densely clustered near the origin. Let

$$h_j = t_j - t_{j-1}, \quad j = 1, \ldots, N; \; h = \max_{1 \leq j \leq N} h_j. \tag{4}$$

It is easy to see that

$$h_j \leq h \leq \frac{rT}{N}, \quad j = 1, \ldots, N$$

and

$$\frac{t_{j+1}}{t_j} \leq 2^r, \quad j = 1, \ldots, N-1. \tag{5}$$

It follows by setting $t = t_i$ $(i \geq 1)$ in (1) that

$$u(t_i) = \int_0^{t_i} \frac{s^{\mu-1}}{t_i^\mu} u(s) \mathrm{d}s + g(t_i). \tag{6}$$

In Euler method, we approximate $u(s)$ on each subinterval $[t_j, t_{j+1}]$ by $u(t_j)$. Define

$$D_j := \int_{t_j}^{t_{j+1}} s^{\mu-1} \mathrm{d}s = \frac{t_{j+1}^\mu - t_j^\mu}{\mu}. \tag{7}$$

Then the numerical scheme is given by

$$u_i^N = g(t_i) + \frac{1}{t_i^\mu} \sum_{j=0}^{i-1} D_j u_j^N, \quad i = 1, 2, \ldots, N, \tag{8}$$

with $u_0^N = u(0)$.

Remark 2.1. In the case $\mu > 1$ or $\mu < 1$, $u(0) = \frac{\mu}{\mu-1} g(0)$ (with $c_0 = 0$); In the case $\mu = 1$, $g(0) \equiv 0$, the value of $u(0)$ needs to be given in advance. Since $h_1 = T/N^r$ and $u \in C^1[0,T]$, we have

$$\max_{s \in [t_0, t_1]} |u(s) - u(t_0)| \leq T \max_{s \in [t_0, t_1]} |u'(s)| \frac{1}{N^r}. \tag{9}$$

Define the truncation error $\delta(N, t_i)$ via

$$u(t_i) = g(t_i) + \frac{1}{t_i^\mu} \sum_{j=0}^{i-1} D_j u(t_j) + \delta(N, t_i), \quad i \geq 1. \tag{10}$$