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# On the convergence of modified Newton methods for solving equations containing a non-differentiable term

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ABSTRACT

We provide a semilocal convergence analysis for certain modified Newton methods for solving equations containing a non-differentiable term. The sufficient convergence conditions of the corresponding Newton methods are often taken as the sufficient conditions for the modified Newton methods. That is why the latter methods are not usually treated separately from the former. However, here we show that weaker conditions, as well as a finer error analysis than before can be obtained for the convergence of modified Newton methods. Numerical examples are also provided.



## 1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of an equation

$$F(x) + G(x) = 0, \quad (1.1)$$

where $F$ is a Fréchet differentiable operator, $G$ is a continuous operator both defined on the same convex subset of a Banach space $X$ with values in a Banach space $Y$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x) = F(x) + G(x)$, where $x$ is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

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We propose the modified Newton methods

$$x_{n+1} = x_n - F'(x_0)^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0) \tag{1.2}$$

or

$$x_{n+1} = x_n - L^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0), L = F'(x_0) + [x_{-1}, x_0; G] \tag{1.3}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x) \in L(X, Y)$, the space of bounded linear operators from $X$ into $Y$ and $[x, y; G]$ is a divided difference of order one for the operator $G$ satisfying

$$[x, y; G](x - y) = G(x) - G(y) \tag{1.4}$$

for all $x \neq y$ [1,2].

Let us also define related Newton methods

$$x_{n+1} = x_n - F'(x_n)^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0) \tag{1.5}$$

or

$$x_{n+1} = x_n - L_n^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0), \; L_n = F'(x_n) + [x_{n-1}, x_n; G]. \tag{1.6}$$

The sufficient convergence conditions for faster Newton methods (1.5) and (1.6) already in the literature [2–21], (see also Remarks 2.3 and 2.6) are also used for slower modified Newton methods (1.2) and (1.3). Here motivated by optimization considerations we show that weaker sufficient conditions for the semilocal convergence of modified Newton methods (1.2) and (1.3) can be obtained, by simply introducing center Lipschitz-type conditions (see (2.1)) instead of the stronger Lipschitz-type conditions (see (2.18)) usually associated with methods (1.2), (1.3), (1.5) and (1.6). Numerical examples are provided where our weaker conditions are satisfied but the ones in the literature are not [2,19–21]. We also note that if our conditions hold but the stronger ones cannot, we can start with slower method (1.2) (or (1.3)) until a certain finite number of steps $N$ at which $x_N$ can be the initial guess of faster method (1.5) (or (1.6)), (since then the stronger hypotheses for Newton method (1.5) (or (1.6)) will then be satisfied). Such a work has already been done by us in [1,22,23] connecting modified Newton method

$$x_{n+1} = x_n - F'(x_0)^{-1}(F(x_n)) \quad (n \geq 0) \tag{1.7}$$

to Newton method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \tag{1.8}$$

or modified Newton method

$$x_{n+1} = x_n - A(x_0)^{-1}(F(x_n)) \quad (n \geq 0) \tag{1.9}$$

to

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad (n \geq 0), \tag{1.10}$$

where $A(x) \in L(X, Y)$ is an approximation to $F'(x)$ (see also [22–24]).

There is an extensive literature for methods (1.2), (1.3) and (1.5)–(1.10). A survey of such results can be found in [1] (see also [2–25]). Iteration (1.5) was first treated in [21]. A finer convergence analysis was later provided in [1,4–6,19,20,22–25]. Qi in [13,14] provided a local as well as a semilocal convergence analysis on $\mathbb{R}^i$ of Newton method (1.8) using directional derivatives, BD-regularity, and locally Lipschitzian functions $F$. A natural damping of Newton method for nonsmooth Newton method (1.8) via the path search was presented in [15]. The q-quadratic convergence was also established in the same reference. Han et al. [9] studied the damped Newton and Gauss methods using directional and Clarke derivatives. Dingguo et al. in [8] studied large size equations on $\mathbb{R}^i$, and also provided a way of controlling the residuals appearing in Newton method (1.8), when $F'$ is replaced by $\nabla F$, the gradient of $F$. A locally convergence analysis was provided in [16,17], where $F : \mathbb{R}^i \to \mathbb{R}^i$ is locally Lipschitz continuous. The super-linear convergence of Newton method (1.8) was shown in [6] by using one sided directionally differentiable operators $F$.

## 2. Semilocal convergence analysis of modified Newton methods (1.2) and (1.3)

We need a result from [20, p 673].

**Lemma 2.1.** *Let $T$ be an operator which is defined on $\overline{U}(x_0, R) = \{x \in X : \|x - x_0\| \leq R\} \subseteq X$ with values in $Y$, and which satisfies a Lipschitz condition*

$$\|T(x) - T(y)\| \leq v(r)\|x - y\|, \quad \text{for all } x, y \in \overline{U}(x_0, r), \text{ and all } r \in [0, R],$$

*for some non-decreasing function $v(r)$ on $[0, R]$. Then, the following hold true:*

$$\|T(x + h) - T(x)\| \leq \gamma(r + \|h\|) - \gamma(r), \quad \text{for all } x \in \overline{U}(x_0, r), \; \|h\| \leq R - r,$$