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Error bound results for generalized D-gap functions of nonsmooth variational inequality problems☆

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ABSTRACT

Solving a variational inequality problem can be equivalently reformulated into solving a unconstraint optimization problem where the corresponding objective function is called a merit function. An important class of merit function is the generalized D-gap function introduced in [N. Yamashita, K. Taji, M. Fukushima, Unconstrained optimization reformulations of variational inequality problems, J. Optim. Theory Appl. 92 (1997) 439–456] and Yamashita and Fukushima (1997) [17]. In this paper, we present new fractional local/global error bound results for the generalized D-gap functions of nonsmooth variational inequality problems, which gives an effective estimate on the distance between a specific point to the solution set, in terms of the corresponding function value of the generalized D-gap function. Numerical examples and a simple application to the free boundary problem are also presented to illustrate the significance of our error bound results.

1. Introduction

The famous variational inequality problem (VIP) occurs in many areas. A representative prototype of the problems we consider throughout this paper is the following:

Definition 1.1. Let F be a mapping from $\mathbb{R}^n$ into itself and let S be a nonempty closed convex subset of $\mathbb{R}^n$. Then the variational inequality problem, denoted by $VI(F, S)$, is to find a vector $x \in S$ such that

$$\langle F(x), y - x\rangle \geq 0 \quad \text{for all } y \in S \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$.

When S is the nonnegative orthant in $\mathbb{R}^n$, $VI(F, S)$ reduces to the nonlinear complementarity problem $NCP(F)$, i.e., finding a vector $x \in \mathbb{R}^n_+$, $F(x) \in \mathbb{R}^n_+$ such that $F(x)^T x = 0$. Variational inequality problems have been widely studied in various fields such as mathematical programming, game theory and economics etc.; see [1,2] and references therein for the background information and motivation of the variational inequality problems covering both smooth and nonsmooth functions. In fact nonsmooth variational problems are quite abundant, see [3–9] for recent developments. Below, let us mention explicitly one of the simplest examples (which is borrowed from [3]).

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Example 1.1. Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Given two positive numbers λ and $p \in (0, 1)$, consider the following free boundary problem

$$\Delta u + \lambda u^p = 0 \quad \text{in } \Omega^+,$$
$$u = 0 \quad \text{in } \Omega_0,$$
$$u = |\nabla u| = 0 \quad \text{on } \Gamma,$$
$$u = 1 \quad \text{on } \partial\Omega$$

where $\Omega^+ = \{z \in \Omega : u(z) > 0\}$, $\Omega_0 = \{z \in \Omega : u(z) = 0\}$, and $\Gamma = \partial\Omega_0 = \partial\Omega^+ \cap \Omega$ are unknown. Using finite element approximation or finite difference approximation, we obtain a nonlinear complementarity problem $NCP(F)$ with $F(x) = Mx + \phi(x)$, where M is a $n \times n$ matrix and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\phi(x) = E \max\{0, x\}^p + q,$$

(here E is an $n \times n$ diagonal matrix with positive diagonals, $\max\{0, x\}^p = (\max\{0, x_1\}^p, \ldots, \max\{0, x_n\}^p)^T$ and q is a vector in $\mathbb{R}^n$). Since $p \in (0, 1)$, it can be verified that φ is not differentiable at the origin and $\lim_{a\to 0, a>0} \|\nabla\varphi(a)\| \to +\infty$. Therefore, we see that the corresponding nonlinear complementarity problem is an example of a nonsmooth (non-Lipschitzian) variational inequality problem.

In recent years, much effort has been made to reformulate the variational inequality problems (VIP) as equivalent optimization problems through the merit functions. Among them, the D-gap functions for VIP are particularly interesting because they cast the VIP as equivalent unconstrained optimization problems. On the other hand, the theory of error bounds provides a useful aid for understanding the connection between the merit function and the actual distance function to the given set and hence provides valuable information about the iterates obtained at the termination of the iterative algorithms. Therefore, it is interesting and useful to investigate the error bound result for the D-gap function. Recently, under the assumption that F is smooth, globally Lipschitz and strongly monotone, Peng and Fukushima [10] gave a fractional global error bound result for D-gap functions. Moreover, Yamashita, Taji and Fukushima [11] extended the D-gap functions to the generalized D-gap functions and thereby gave an corresponding error bound results for generalized D-gap functions. Very recently, under the assumption that F is smooth and strongly monotone, Qu, Wang, Zhang [12] extended the work of [10] and presented a local error bound result based on the generalized D-gap functions (for other closed related work, see [13, 14]). However, all the error bound results for D-gap functions mentioned above are only for smooth and strongly monotone function F. In this paper, under the weaker assumption that F is continuous, locally ξ-monotone and coercive, we establish some fractional global/local error bound results for generalized D-gap functions.

This paper is organized as follows: In Section 2, we review some basic results that will be used in this paper. In Section 3, we collect some important properties of the generalized D-gap functions. In Section 4, we establish fractional local/global error bound results under appropriate conditions. Finally, as an application, we provide a global error bound result for the nonsmooth complementarity problem which arises from the free boundary problem.

2. Preliminaries

We first review some concepts related to the nonsmooth analysis and variational inequality problem. Throughout this paper, the n-dimensional Euclidean real vector space will be denoted by $\mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$, the inner product between x and y is defined by $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ where x_i (resp. y_i) is the ith coordinate of x (resp. y). The norm arising from the inner product is the usual Euclidean norm on the respective $\mathbb{R}^n$ space denoted by $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$. The open unit ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$ is denoted by $B(0; 1)$ and the closed unit ball $\{x \in \mathbb{R}^n : \|x\| \le 1\}$ is denoted by $\overline{B}(0; 1)$. For an arbitrary subset C of $\mathbb{R}^n$, we denote the interior, closure, convex hull and topological boundary of it by $\operatorname{int} C$, $\overline{C}$, $\operatorname{conv}(C)$ and ∂C, respectively.

Definition 2.1. Let $m, n \in \mathbb{N}$. A vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be (locally) Lipschitz near $x_0 \in \mathbb{R}^n$ if there exist two positive constants δ and L such that

$$\|F(x_1) - F(x_2)\| \le L\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in B(x_0; \delta).$$

The vector valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally Lipschitz on an open subset D of $\mathbb{R}^n$ if F is (locally) Lipschitz near any point $x_0 \in D$. Moreover, we say F is globally Lipschitz (with modulus L) on an open subset D, if there exists a positive constant L such that

$$\|F(x_1) - F(x_2)\| \le L\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in D.$$

Definition 2.2. Let S be a closed subset of $\mathbb{R}^n$. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) monotone on S if for all $x, y \in S$, $\langle F(x) - F(y), x - y\rangle \ge 0$;
(b) strictly monotone on S if for all $x, y \in S$ with $x \ne y$, $\langle F(x) - F(y), x - y\rangle > 0$;