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Common fixed point theorems for non-self-mappings in metric spaces of hyperbolic type

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ABSTRACT

In this paper, the concept of a pair of non-linear contraction type mappings in a metric space of hyperbolic type is introduced and the conditions guaranteeing the existence of a common fixed point for such non-linear contractions are established. Presented results generalize and improve some of the known results. An example is constructed to show that our theorems are genuine generalizations of the main theorems of Assad, Ćirić, Khan et al., Rhoades and Imdad and Kumar. One of the possible applications of our results is also presented.

1. Introduction

Fixed point theorems for contraction self-mappings have find applications in diverse disciplines of mathematics, engineering and economics. In convex spaces occur cases where the involved function is not necessarily a self-mapping of a closed subset. Assad [1] and Assad and Kirk [2] first studied non-self-contraction mappings in a metric space (X, d), metrically convex in the sense of Menger (that is, for each x, y in X with $x \neq y$ there exists z in X, $x \neq z \neq y$, such that $d(x, z) + d(z, y) = d(x, y)$). In recent years, this technique has been developed and fixed and common fixed points of non-self-mappings have been studied by many authors [3–14]. Some of the obtained results have find applications (c.f. [2,14–16]). In numerical mathematics, a restricted condition $T(\partial K) \subseteq K$ is especially favorable instead of $T(K) \subseteq K$, where K is a closed subset of X, $T : K \to X$ and ∂K is the boundary of K.

In an attempt to generalize a theorem of Assad [1] and Assad and Kirk [2], Rhoades [13] proved the following result in a Banach space.

Theorem 1. *Let X be a Banach space, K a non-empty closed subset of X and $T : K \to X$ a mapping of K into X satisfying the condition*

$$d(Tx, Ty) \leq h \max\left\{\frac{d(x, y)}{2}, d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q}\right\} \tag{1}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$ and T has the additional property that for each $x \in \partial K$, the boundary of K, $Tx \in K$, then T has a unique fixed point.

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Recently Imdad and Kumar [10] generalized the result of Rhoades [13] and Ćirić [3]. They proved the following theorem.

Theorem 2. *Let X be a Banach space, K a non-empty closed subset of X and $F, T : K \to X$ two mappings satisfying the condition*

$$d(Fx, Fy) \leq h \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \frac{d(Tx, Fy) + d(Ty, Fx)}{q}\right\} \tag{2}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$ and

(i) $\partial K \subseteq TK$, $FK \cap K \subset TK$,

(ii) $Tx \in \partial K \Longrightarrow Fx \in K$, *and*

(iii) TK *is closed in* X.

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z remains a unique common fixed point of F and T.

Recall (see [17]) that a pair of mappings (F, T), defined on a non-empty set S, is said to be coincidentally commuting, if $Tx = Fx \Longrightarrow FTx = TFx$; $x \in S$.

The purpose of this paper is to introduce the concept of a new pair of non-linear contractive type non-self-mappings which satisfy a new contractive condition, weaker than (2) and to prove common fixed point theorems in metric spaces of hyperbolic type. Our theorems generalize the main theorems of Assad [1], Ćirić [3], Khan et al. [12], Rhoades [13] and Imdad and Kumar [10] in many aspects. An example is constructed to show that our results are genuine generalizations of the known results from this area. One of the possible applications of our results is also presented.

2. Main results

Throughout our consideration we suppose that (X, d) is a convex metric space which contains a family L of metric segments (isometric images of real line segments) such that

(a) each two points x, y in X are end points of exactly one member $seg[x, y]$ of L, where

$$seg[x, y] = \{z \in X : d(z, x) = \lambda d(x, y) \text{and } d(z, y) = (1 - \lambda)d(x, y); \lambda \in [0, 1]\},$$

and

(b) if u, x, y in X and if $z \in seg[x, y]$ satisfies $d(x, z) = \lambda d(x, y)$ for any $\lambda \in [0, 1]$, then

$$d(u, z) \leq (1 - \lambda)d(u, x) + \lambda d(u, y). \tag{3}$$

A space of this type is said to be *a metric space of hyperbolic type* (Takahashi [18] uses the term *a convex metric space*). This class includes all normed linear spaces, as well as all spaces with hyperbolic metric (see [19] for a discussion). For instance, if X is a Banach space, then

$$seg[x, y] = \{(1 - \lambda)x + \lambda y : 0 \leq \lambda \leq 1\}.$$

A linear space with a translation invariant metric satisfying

$$d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0)$$

is also a metric space of hyperbolic type. There are many other examples but we consider these as paradigmatic.

Now we shall prove a common fixed point theorem for a new pair of non-linear contraction type mappings in a metric space of hyperbolic type. Recall that the concept of a non-linear contraction was introduced and studied in [20], and that some applications of non-linear contractions was considered in [21].

Theorem 3. *Let X be a metric space of hyperbolic type, K a non-empty closed subset of X and ∂K the boundary of K. Let ∂K be non-empty and let $T : K \to X$ and $F : K \cap T(K) \to X$ be two non-self-mappings satisfying the following conditions:*

$$d(Fx, Fy) \leq \varphi\left(\max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Fy) + d(Ty, Fx)}{3}\right\}\right), \tag{4}$$

for all x, y in X, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a real function which has the following properties:

$\varphi(t+) < t$ for $t > 0$ and $\varphi(t)$ is non-decreasing.

Suppose that F and T have the additional properties:

(i) $\partial K \subseteq TK$;

(ii) $FK \cap K \subseteq TK$;

(iii) $Tx \in \partial K \Rightarrow Fx \in K$;

(iv) $K \cap T(K)$ *is complete.*

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z is a unique common fixed point of F and T.