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Padé approximation and Apostol–Bernoulli and Apostol–Euler polynomials

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ABSTRACT

Using the Padé approximation of the exponential function, we obtain recurrence relations between Apostol–Bernoulli and between Apostol–Euler polynomials. As applications, we derive some new lacunary recurrence relations for Bernoulli and Euler polynomials with gap of length 4 and lacunary relations for Bernoulli and Euler numbers with gap of length 6.

1. Introduction

Classical Bernoulli and Euler polynomials play a fundamental role in various branches of mathematics including combinatorics, number theory and special functions. They are usually defined by means of the following generating functions

$$\frac{t}{e^t-1}e^{xt}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!},\quad (|t|<2\pi), \tag{1.1}$$

$$\frac{2}{e^t+1}e^{xt}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!},\quad (|t|<\pi). \tag{1.2}$$

The first polynomials are

$$B_0(x)=1,\qquad E_0(x)=1,$$
$$B_1(x)=x-1/2,\qquad E_1(x)=x-1/2,$$
$$B_2(x)=x^2-x+1/6,\qquad E_2(x)=x^2-x.$$

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From (1.1) and (1.2), it is easily proved that

$$B_n(1-x) = (-1)^n B_n(x), \qquad E_n(1-x) = (-1)^n E_n(x),$$
$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad E_n(x+1) + E_n(x) = 2x^n,$$
$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \qquad E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

The classical Bernoulli numbers B_n and the classical Euler numbers E_n are given by $B_n := B_n(0)$ and $E_n := 2^n E_n(1/2)$, respectively.

In 2000, T. Agoh [1] proved a general linear recurrence relation between Bernoulli and Euler polynomials.

More recently, in [2], Chen and Sun made use of Zeilberger's algorithm to prove most of the existing recurrence relations for Bernoulli and Euler polynomials. They also derived two new identities which are particular cases of the main theorem of this paper.

Some analogues of the classical Bernoulli polynomials were introduced by Apostol in order to evaluate the Hurwitz–Lerch zeta function:

$$\phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}.$$

See [3] and also the recent book [4].

We begin by recalling here Apostol's definition as follows:

Definition 1 (*Apostol, [3]*). The Apostol–Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$ in the variable x are defined by means of the following generating function:

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda) \frac{t^k}{k!} \tag{1.3}$$
$$(|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\log \lambda| \text{ when } \lambda \neq 1).$$

Using the Gaussian hypergeometric functions, Luo [5] found formulas for $\mathcal{B}_k(x;\lambda)$ and in [6], Boyadzhiev found the relations of the Apostol–Bernoulli functions with the Euler polynomials and the derivative polynomials for the cotangent function.

Recently, Luo and Srivastava introduced the Apostol–Bernoulli polynomials of higher order (also called generalized Apostol–Bernoulli polynomials):

Definition 2 (*Luo and Srivastava, [7]*). The Apostol–Bernoulli polynomials $\mathcal{B}_k^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} \mathcal{B}_k^{(\alpha)}(x;\lambda) \frac{t^k}{k!}, \tag{1.4}$$
$$(|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\log \lambda| \text{ when } \lambda \neq 1).$$

According to the definition, by setting $\alpha = 1$, we obtain the Apostol–Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$. Moreover, we call $\mathcal{B}_k(\lambda) := \mathcal{B}_k(0;\lambda)$ the Apostol–Bernoulli numbers.

Explicit representation of $\mathcal{B}_k^{(\alpha)}(x;\lambda)$ in terms of a generalization of the Hurwitz–Lerch zeta function can be found in [8].

In the paper [9] submitted in 2004, Luo introduced Apostol–Euler polynomials of higher order α:

Definition 3 (*Luo, [9]*). The Apostol–Euler polynomials $\mathcal{E}_k^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} \mathcal{E}_k^{(\alpha)}(x;\lambda) \frac{t^k}{k!}, \quad (\lambda \neq -1,\ |t| < |\log(-\lambda)|). \tag{1.5}$$

The Apostol–Euler polynomials $\mathcal{E}_k(x;\lambda)$ are given by $\mathcal{E}_k(x;\lambda) := \mathcal{E}_k^{(1)}(x;\lambda)$. The Apostol–Euler numbers $\mathcal{E}_k(\lambda)$ are given by $\mathcal{E}_k(\lambda) := 2^k \mathcal{E}_k(\frac{1}{2};\lambda)$.

Some relations between Apostol–Bernoulli and Apostol–Euler polynomials of order α can be found in [10]. For more results on these polynomials, the readers are referred to [11–13].