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# Oversampling and reconstruction functions with compact support

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ABSTRACT

Assume that a sequence of samples of a filtered version of a function in a shift-invariant space is given. This paper deals with the existence of a sampling formula involving these samples and having reconstruction functions with compact support. This is done in the light of the generalized sampling theory by using the oversampling technique. A necessary and sufficient condition is given in terms of the Smith canonical form of a polynomial matrix. Finally, we prove that the aforesaid oversampled formulas provide nice approximation schemes with respect to the uniform norm.



## 1. Statement of the problem

Let $V_\varphi$ be a shift-invariant space in $L^2(\mathbb{R})$ with stable generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi := \left\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}).$$

Nowadays, sampling theory in shift-invariant spaces is a very active research topic (see, for instance, [1–4,9,17,18] and the references therein) since an appropriate choice for the generator $\varphi$ (for instance, a B-spline) eliminates most of the problems associated with the classical Shannon's sampling theory [16].

Suppose that a linear time-invariant system $\mathcal{L}$ is defined on $V_\varphi$. Under suitable conditions, Unser and Aldroubi [3,15] have found sampling formulas allowing the recovering of any function $f \in V_\varphi$ from the sequence of samples $\{(\mathcal{L}f)(n)\}_{n\in\mathbb{Z}}$. Concretely, they proved that for any $f \in V_\varphi$,

$$f(t) = \sum_{n\in\mathbb{Z}} \mathcal{L}f(n) S(t-n), \quad t \in \mathbb{R}, \tag{1}$$

where the sequence $\{S(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V_\varphi$. Even when the generator $\varphi$ has compact support, rarely the same occurs with the reconstruction function $S$ in formula (1). Recall that a reconstruction function $S$ with compact support in (1) implies low computational complexities and avoids truncation errors. A way to overcome this difficulty is to use the oversampling technique, i.e, to take samples with a sampling period $T < 1$. This is the main goal in this paper: Assuming that

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the generator $\varphi$ and the impulse response of the linear system $\mathcal{L}$ have compact support, we derive stable sampling formulas which allow us to recover any $f \in V_\varphi$ from the samples $\{(\mathcal{L}f)\,(T_s n)\}_{n\in\mathbb{Z}}$, where the sampling period is $T_s := (s-1)/s < 1$ for some $s \in \{2, 3, \ldots\}$. This is done in Sections 2 and 3 in the light of the generalized sampling theory obtained in [10] by following an idea of Djokovic and Vaidyanathan in [9].

For the sake of notational ease we have assumed that only samples from one linear time-invariant system $\mathcal{L}$ are available. Analogous results are still valid in the case of several systems. In [7], a different but related question is studied: Roughly speaking, assuming that $\varphi$ has compact support a system $\mathcal{L}$ with impulse response compactly supported is found in order to recover any function in $V_\varphi$ by using the generator itself as the reconstruction function.

Besides, shift-invariant spaces are important in a number of areas of analysis. Many spaces, encountered in approximation theory and in finite element analysis, are generated by the integer shifts of a function $\varphi$. Shift-invariant spaces also play a key role in the construction of wavelets [13]. In each of these applications, one is interested in how well a general smooth function (in a potential Sobolev space) can be approximated by the elements of the scaled spaces $\sigma_h V_\varphi := \{f(\cdot/h) \,:\, f \in V_\varphi\}$ (see [5] and the references therein). A cornerstone in this theory are the Strang–Fix conditions for the generator $\varphi$ [14].

On the other hand, as pointed out by Lei et al. in [12], there are many ways to construct approximation schemes associated with shift-invariant spaces. Among them, they cite cardinal interpolation, quasi-interpolation, projection and convolution (see the references in [12]). They unify these approaches in a systematic way by viewing all as special cases of the approximation scheme induced by an integral operator. Borrowing a result in [12], in Section 4 we prove that the oversampled formulas with compactly supported reconstruction functions obtained in Section 3 give "good" approximation schemes with respect to the sup norm.

## 2. A sampling formula in the oversampling setting

From now on, the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for the shift-invariant space

$$V_\varphi := \left\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V_\varphi$. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V_\varphi$ if and only if

$$0 < \|\Phi\|_0 \le \|\Phi\|_\infty < \infty,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ in $(0, 1)$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [6, p. 143].

We assume throughout the paper that the functions in the shift-invariant space $V_\varphi$ are continuous on $\mathbb{R}$. Equivalently, the generator $\varphi$ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [18]). Thus, any $f \in V_\varphi$ is defined as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$ on $\mathbb{R}$. Besides, $V_\varphi$ is a reproducing kernel Hilbert space where convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$ (see [10]).

The space $V_\varphi$ is the image of $L^2(0, 1)$ by means of the isomorphism $\mathcal{T}_\varphi : L^2(0, 1) \to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0, 1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for $V_\varphi$. Namely, for each $F \in L^2(0, 1)$ the function $\mathcal{T}_\varphi F \in V_\varphi$ is given by

$$(\mathcal{T}_\varphi F)(t) := \sum_{n\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi i n \cdot} \rangle_{L^2(0,1)} \varphi(t-n), \quad t \in \mathbb{R}. \tag{2}$$

Suppose that $\mathcal{L}$ is a linear time-invariant system defined on $V_\varphi$ of one of the following types (or a linear combination of both):

(a) The impulse response $\mathsf{h}$ of $\mathcal{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\varphi$ we have

$$(\mathcal{L}f)\,(t) := [f * \mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)\mathrm{d}x, \quad t \in \mathbb{R}.$$

(b) $\mathcal{L}$ involves samples of the function itself, i.e., $(\mathcal{L}f)(t) = f(t+d)$, $t \in \mathbb{R}$, for some constant $d \in \mathbb{R}$.

For a fixed $s \in \{2, 3, \ldots\}$, consider $T_s = (s-1)/s < 1$. The first goal is to recover any function $f \in V_\varphi$ by using a frame expansion involving the samples $\{(\mathcal{L}f)(T_s n)\}_{n\in\mathbb{Z}}$. This can be done in the light of the generalized sampling theory developed in [10]. Indeed, since the sampling points $T_s n$, $n \in \mathbb{Z}$, can be expressed as

$$\{T_s n\}_{n\in\mathbb{Z}} = \{(s-1)n + (j-1)T_s\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s},$$

the initial problem is equivalent to the recovery of $f \in V_\varphi$ from the samples

$$\{\mathcal{L}_j f\,((s-1)n)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$$

where the linear time-invariant systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are defined by

$$(\mathcal{L}_j f)(t) := (\mathcal{L}f)\,[t + (j-1)T_s]\,, \quad t \in \mathbb{R}.$$