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ScienceDirect
Journal of Computational and Applied Mathematics 221 (2008) 114–131
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Function-valued Padé-type approximant via the formal orthogonal polynomials and its applications in solving integral equations☆

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Received 24 May 2007; received in revised form 19 September 2007

Abstract

A kind of function-valued Padé-type approximant via the formal orthogonal polynomials (FPTAVOP) is introduced on the polynomial space and an algorithm is sketched by means of the formal orthogonal polynomials. This method can be applied to approximate characteristic values and the corresponding characteristic function of Fredholm integral equation of the second kind. Moreover, theoretical analyses show that FPTAVOP method is the most effective one for accelerating the convergence of a sequence of functions. In addition, a typical numerical example is presented to illustrate when the estimates of characteristic value and characteristic function by using this new method are more accurate than other methods.

Keywords: Padé approximant; Formal orthogonal polynomial; Neumann series; Fredholm integral equation; Characteristic value; Convergence acceleration

1. Introduction

Consider a Fredholm integral equation of the second kind

$$f(x,\lambda) = g(x) + \lambda \int_a^b K(x,y) f(y,\lambda) \mathrm{d}y, \quad (x,y) \subset [a,b] \times [a,b],$$

where $g(x) \in L^2[a,b]$ and $K(x,y)$ is an L_2 kernel which are defined in $[a,b]$ and $[a,b] \times [a,b]$, respectively.

The technique utilized for solving the integral equation is based on successive substitution, which is an iterative procedure, yielding a sequence of a approximations leading to an infinite power series solution. So we turn to consider the generating function $f(x,\lambda)$ of this kind of series of functions given by

$$f(x,\lambda) = \sum_{i=0}^{\infty} c_i(x) \lambda^i, \tag{1}$$

☆ The work is supported by the National Natural Science Foundation of China (10271074) and by the Special Funds for Major Specialities of Shanghai Education Committee.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.008

in which $c_i(x) \in L_2[a, b]$ are given by the successive substitution and $[a, b]$ is the domain of definition of $c_i(x)$ in this natural sense. We suppose that $f(x, \lambda)$ is holomorphic as a function of λ at the origin $\lambda = 0$. Then $f(x, \lambda)$ in (1) converges for values of $|\lambda|$ which are sufficiently small.

Many methods estimating the characteristic values and characteristic function of (1) have been derived in the previously published papers, such as the classic Padé approximant method (CPA) [7,8], the generalized inverse, functional-valued Padé approximant method (GIPA) [7,8], the modified Padé approximant method (MPA) [10], the squared Padé approximant method (SPA) [10], the integral Padé approximant method (IPA) [10], the function-valued Padé-type approximant method (FPTA) [13], and the ε-algorithm of function-valued GIPA [11,12].

The major drawback of the method of CPA [7,8] is the use of the minimal sensitivity principle [1,2] and the presence of superfluous zeros in the denominator. Thus we have to assign a particular value of x in the Neumann series in order to obtain the estimate of the characteristic values. Graves-Morris introduced the method of function-valued GIPA [7,8] by using the generalized inverse which had been defined in the case of vector-valued GIPA [9] and a solution in which the numerator function and the denominator polynomial can be represented by determinants of the same dimension. Thukral investigated on the basis of this similar principle and Padé-type approximants introduced in [4,5], which means that the denominator polynomial of the rational approximant is arbitrarily prescribed (whereas, in the classical Padé approach the denominator is left free in order to achieve the maximal order of interpolation), and introduced three methods of Padé-type approximant, namely IPA [10], MPA [10] and SPA [10]. The construction of the denominator of the MPA [10] method is simply obtained by integrating each of the cofactors in the determinant of the denominator polynomial of the classical Padé approximant method. The denominator of the SPA [10] method was constructed by squaring each of the cofactor in the determinant of the denominator polynomial of the classical Padé approximant method and then integrating these new cofactors. The construction of the denominator of the IPA [10] method is obtained by combining the coefficients of the generating function as cofactors in the determinant of the denominator polynomial. The method of FPTA [13] was constructed by a an approach similar to that of the method of IPA [10]. The construction of the ε-algorithm of function-valued GIPA [11,12] was the same as that of ε-algorithm of scalar Padé approximants, just using the generalized inverse in order to keep the denominator polynomial of GIPA [7,8] having the only one parameter of λ.

In this paper we represent a new method, namely function-valued Padé-type approximant via the formal orthogonal polynomials (FPTAVOP), for summing the series of function (1). This method overcomes all essential difficulties encountered in the previous studies and is simpler and more effective for obtaining the characteristic values and the characteristic function than all those methods we have mentioned above.

The remainder of this paper is organized as follows. In Section 2, we mainly derive the method of FPTAVOP. We extend scalar Padé-type approximant to function-valued Padé-type approximant in Section 2.1, introduce the definition of FOPs with respect to $f(x, \lambda)$ and their three-term recurrence relationship and sketch an algorithm to compute the FOPs with respect to $f(x, \lambda)$ in Section 2.2, combine function-valued Padé-type approximant and the FOPs with respect to $f(x, \lambda)$ together to construct FPTAVOP and sketch a main algorithm to compute FPTAVOP in Section 2.3, and finally, in Section 2.4, we present a numerical example to show the effectiveness of FPTAVOP for solving integral equation. In Section 3, we first give the constructions of the methods of GIPA, ε-algorithm of GIPA, IPA, MPA, CPA, SPA and FPTA in the Sections 3.1 and 3.2, and then apply these methods to a typical numerical example in Section 3.3, and finally, in Section 3.4, we make a comparison of the estimates of the characteristic values of the integral equation derived using all these different methods. The concluding remarks are given in Section 4.

2. Function-valued Padé-type approximant via the formal orthogonal polynomials

In this section, we derive a new approach via the formal orthogonal polynomials, i.e., FPTAVOP to approximate the function-valued power series $f(x, \lambda)$ in (1).

2.1. Function-valued Padé-type approximant (FPTA)

Let $c : \mathbf{P} \to \mathbf{C}$ be a linear functional on the polynomial space $\mathbf{P}$, and define it by

$$c(t^i) = c_i(x), \quad i = 0, 1, \dots \tag{2}$$