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Existence of triple positive solutions for a third-order three-point boundary value problem

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Abstract

In this paper we investigate the existence of triple positive solutions for the nonlinear third-order three-point boundary value problem

$$u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1,$$
$$u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \quad u''(1) = 0,$$

where $\delta \in (0, 1)$, $\eta \in [1/2, 1)$ are constants. $f : [0, 1] \times [0, \infty) \times R^2 \to [0, \infty)$, $q : (0, 1) \to [0, \infty)$ are continuous. First, Green's function for the associated linear boundary value problem is constructed, and then, by using a fixed-point theorem due to Avery and Peterson, we establish results on the existence of triple positive solutions to the boundary value problem.

MSC: 34B15

Keywords: Positive solutions; Third-order three-point boundary value problem; Fixed-point theorem

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, the existence and multiplicity of positive solutions for nonlinear third-order ordinary differential equations with a three-point boundary value problem (BVP for short) have been studied by several authors. An interest in triple solutions evolved from the Leggett–Williams multiple-fixed-point theorem [11]. And lately, two triple-fixed-point theorems due to Avery [5] and Avery and Peterson [6], have been applied to obtain triple solutions of certain three-point boundary value problems for third-order ordinary differential equations. For example, Anderson [1] proved that there exist at least three positive solutions to the BVP

$$-x'''(t) + f(x(t)) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(t_2) = x''(1) = 0,$$

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where $f : R \to R$ is continuous, f is nonnegative for $x \geq 0$, and $1/2 \leq t_2 < 1$. Bai and Fei [7] obtained the sufficient conditions for the existence of at least three positive solutions for the third-order three-point generalized right focal problem

$$x''' = q(t)f(t, x, x', x'''), \quad t_1 < t < t_3,$$
$$x(t_1) = x'(t_2) = 0, \qquad \eta x(t_3) + \delta x''(t_3) = 0,$$

where $f \in C([t_1, t_3] \times [0, \infty) \times R^2, [0, \infty))$, $q \in C((t_1, t_3), [0, \infty))$ and does not vanish identically on any subinterval of (t_1, t_3). Furthermore, $0 < \int_{t_1}^{t_2} q(s)\mathrm{d}s$, $\int_{t_2}^{t_3} q(s)\mathrm{d}s < +\infty$. $\eta \geq 0, \delta > 0, k := 2\delta + \eta(t_3 - t_1)(t_3 - 2t_2 + t_1) > 0, t_1 < t_2 < t_3$ are real numbers with $t_2 - t_1 > t_3 - t_2$. For other existence results for third-order three-point BVP, one may see [2–4,8–10,12–16,18,19] and the references therein.

Motivated greatly by the above-mentioned works, in this paper we will consider the existence of multiple positive solutions (at least three) to the BVP

$$u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \qquad u''(1) = 0, \tag{1.2}$$

where $\delta \in (0, 1)$, $\eta \in [1/2, 1)$ are constants, $a : (0, 1) \to [0, \infty)$ and $f : [0, 1] \times [0, \infty) \times R \times R \to [0, \infty)$ are continuous. Here, by a positive solution of the BVP we mean a function $u^*(t)$ which is positive on $(0, 1)$ and satisfies differential equation (1.1) and the boundary conditions (1.2). Therefore, our positive solutions are nontrivial ones. The methods used in our work will depend on an application of a fixed-point theorem due to Avery and Peterson [6] which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis is put on the nonlinear term involved with all lower-order derivatives explicitly. The paper is organized as follows. In Section 2, we present some notation and lemmas. In Section 3, we give the main results.

2. Preliminaries

In this section, we present some notation and lemmas that will be used in the proof our main results.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a *cone* of E if it satisfies the following two conditions:

(1) $x \in K$, $\lambda > 0$ implies $\lambda x \in K$;
(2) $x \in K$, $-x \in P$ implies $x = 0$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. Suppose K is a cone in a Banach space E. The map α is a nonnegative continuous concave functional on K provided α: $K \to [0, \infty)$ is continuous and

$$\alpha(rx + (1 - r)y) \geq r\alpha(x) + (1 - r)\alpha(y)$$

for all $x, y \in K$ and $r \in [0, 1]$. Similarly, we say the map β is a nonnegative continuous concave functional on K provided β: $K \to [0, \infty)$ is continuous and

$$\beta(rx + (1 - r)y) \leq r\beta(x) + (1 - r)\beta(y)$$

for all $x, y \in K$ and $r \in [0, 1]$.

Let γ and θ be nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and φ a nonnegative continuous functional on K.

For positive real numbers a, b, c, and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in K \mid b \leq \alpha(x), \gamma(x) \leq d\},$$
$$P(\gamma, \theta, b, c, d) = \{x \in K \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$