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Multirate Runge–Kutta schemes for advection equations

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ABSTRACT

Explicit time integration methods can be employed to simulate a broad spectrum of physical phenomena. The wide range of scales encountered lead to the problem that the fastest cell of the simulation dictates the global time step. Multirate time integration methods can be employed to alter the time step locally so that slower components take longer and fewer time steps, resulting in a moderate to substantial reduction of the computational cost, depending on the scenario to simulate [S. Osher, R. Sanders, Numerical approximations to nonlinear conservation laws with locally varying time and space grids, Math. Comput. 41 (1983) 321–336; H. Tang, G. Warnecke, A class of high resolution schemes for hyperbolic conservation laws and convection-diffusion equations with varying time and pace grids, SIAM J. Sci. Comput. 26 (4) (2005) 1415–1431; E. Constantinescu, A. Sandu, Multirate timestepping methods for hyperbolic conservation laws, SIAM J. Sci. Comput. 33 (3) (2007) 239–278]. In air pollution modeling the advection part is usually integrated explicitly in time, where the time step is constrained by a locally varying Courant–Friedrichs–Lewy (CFL) number. Multirate schemes are a useful tool to decouple different physical regions so that this constraint becomes a local instead of a global restriction. Therefore it is of major interest to apply multirate schemes to the advection equation. We introduce a generic recursive multirate Runge–Kutta scheme that can be easily adapted to an arbitrary number of refinement levels. It preserves the linear invariants of the system and is of third order accuracy when applied to certain explicit Runge–Kutta methods as base method.

1. Introduction

Modern air pollution models can be used to simulate the evolution of concentrations of contaminants in the atmosphere. The relevant processes can be described by partial differential equations (PDE) of advection-diffusion-reaction type which can be efficiently approximated by conservative high order spatial discretization methods with implicit/explicit time integration.

For the advection equation

$$\frac{\partial}{\partial t}c + \frac{\partial}{\partial x}(uc) = 0, \tag{1}$$

describing transport of contaminants in air pollution models, explicit Runge–Kutta (ERK) time integration methods have proven to be very efficient. All of these methods have in common that stability requirements limit the global time step to

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be smaller than some critical value proportional to the ratio of grid size and the magnitude of the wind speed for each cell [2,7,14].

In realistic scenarios there are usually regions of interest such as urban areas which have to be examined more closely than the surrounding area, e.g. woodland or agricultural regions. Therefore the spatial grid in these regions is refined [3,4]. Additionally even for equidistant grids the wind speed may vary considerably across the entire domain. Thus the smallest cell, or more exactly the cell with the smallest characteristic time determines the global time step. Multirate time integration methods can be employed to adapt the time step locally so that slower components take longer and fewer time steps, resulting in a moderate to substantial reduction of the computational cost, depending on the scenario to simulate [5].

The multirate scheme we shall introduce is intended to be implemented in a complex three-dimensional weather model. This model shall also employ local refinement strategies and must allow for strongly varying wind speeds. Cell size ratios up to 1:16 and wind speeds of some m/s at ground level in contrast to approximately 100 m/s in large altitudes may lead to characteristic times per cell varying by several orders of magnitude. Therefore the use of a multirate time integration scheme is crucial for efficient computation.

Multirate approaches have been developed since the early 1980s. Osher and Sanders [11] presented a scheme allowing multirate Euler steps. More current approaches allow for a wide variety of multirate schemes by proposing generic multirate methods based on traditional ERKs. In 2005 Tang and Warnecke [12] proposed different multirate schemes which can be generalized to support arbitrary ERKs as base method. The resulting schemes however are not mass preserving. The approach of Constantinescu and Sandu [2] yields multirate ERK schemes which are mass preserving and at most of second order accuracy.

The approach discussed in this paper borrows ideas from an implicit–explicit splitting scheme introduced in [10], where explicit Runge–Kutta methods are combined with an arbitrary implicit time integrator. In contrast to the methods mentioned above this new method is based on a right-hand side splitting and not on a splitting by components. Applied to the discretized advection equation this means that it is based on a splitting of the fluxes instead of flux differences per cell. The new method is called *Recursive Flux Splitting Multirate* (RFSMR).

The main part of this paper, Section 2, is dedicated to the introduction and order analysis of RFSMR. In particular we shall show that the scheme is second order accurate for second order base methods. Furthermore we prove that the third order methods from Knoth and Wolke [10] can be extended to a third order method in the new context. The order analysis will be carried out in the context of partitioned Runge–Kutta methods (PRK). Afterwards, in Section 3 we will present spatial discretizations of the advection equation. A decomposition approach beneficial for the new schemes will be outlined in the context of block structured grids. In Section 4 we will perform numerical tests affirming both stability and accuracy as well as efficiency of this novel class of multirate schemes.

2. Explicit Runge–Kutta multirate scheme based on a right-hand side splitting

The multirate scheme is derived from a splitting of the right-hand side of the differential equation in two parts as follows

$$w' = F(w) + G(w), \qquad w(0) = w_0, \quad w \in \mathbb{R}^N. \tag{2}$$

Starting point is an implicit–explicit (IMEX) integration method introduced in [10] for the efficient solution of advection-diffusion-reaction equations in air pollution. For air pollution models the G term then represents the non-stiff advection part which can be solved using explicit methods. Opposed to this the F term represents the stiff diffusion-reaction part which must be solved using implicit methods. In the cited IMEX scheme the non-stiff, explicit part is integrated by an explicit Runge–Kutta method, the integration method for the stiff part is undetermined. For the theoretical analysis of the method it is assumed that this integration is carried out exactly. In [10] the IMEX method applied to (2) is presented in a generalized Butcher-like tableau. For a given time step Δt the method reads as follows:

$$w_1 = w(t_n), \tag{3}$$

$$w_i = v_i(\tilde{c}_i \Delta t) \quad \text{with} \tag{4}$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}\tau} = \frac{1}{\tilde{c}_i} r_i + F(v_i), \quad \tau \in [0, \tilde{c}_i \Delta t], \qquad v_i(0) = w_{i-1}, \quad i = 2, \ldots, s+1 \tag{5}$$

$$r_i = \sum_{j=1}^{i-1} \tilde{a}_{ij} G(w_j), \tag{6}$$

$$w(t_n + \Delta t) = w_{s+1}, \tag{7}$$

assuming that in each stage the underlying differential equation (5) with the non-discretized implicit term F is integrated exactly. If the implicit term $F \equiv 0$ we obtain the underlying classic explicit Runge–Kutta method which we will call the *outer* method:

$$w_1 = w(t_n) \tag{8}$$