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An elliptic system involving a singular diffusion matrix

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ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a bounded domain. In this work we are interested in finding a renormalized solution to the following elliptic system

$$\begin{cases} -\mathrm{div}[\mathcal{A}_1(u_2)\,\nabla u_1] = f, & \text{in } \Omega \\ -\mathrm{div}[\mathcal{A}_2(u_2)\,\nabla u_2] + g(u_2) = \mathcal{A}_3(u_2)\,\nabla u_1\,\nabla u_1, & \text{in } \Omega, \end{cases} \tag{1}$$

where the diffusion matrix $\mathcal{A}_2$ blows up for a finite value of the unknown, say $u_2 = s_0 < 0$. We also consider homogeneous Dirichlet boundary conditions for both u_1 and u_2. In these equations, u_1 is an N-dimensional magnitude, whereas u_2 is scalar; $\mathcal{A}_2 : \Omega \times (s_0, +\infty) \mapsto \mathbb{R}^N$ is a semilinear coercive operator. The symmetric part of the matrix $\mathcal{A}_3$ is related to the one of $\mathcal{A}_1$. Nevertheless, the behaviour of these coefficients is assumed to be fairly general. Finally, $f \in H^{-1}(\Omega)^N$, and $g : \Omega \times (s_0, +\infty) \mapsto \mathbb{R}$ is a Carathéodory function satisfying the sign condition.

Due to these assumptions, the framework of renormalized solutions for problem (1) is used and an existence result is then established.

1. Introduction

The analysis of system (1) is motivated by the steady state of the so-called k–ε turbulence model. This model was first introduced by Launder and Jones in 1972 [14]. In the k–ε turbulence model, the eddy viscosity coming from the averaged Navier–Stokes equations is written in terms of two new scalar variables: k, the turbulent kinetic energy, and ε, the dissipation of k. In the incompressible case, this model takes the following form

$$\frac{\partial w}{\partial t} + (w\cdot\nabla)w - \mathrm{div}\left[\left(\nu + c_\mu \frac{k^2}{\varepsilon}\right)\nabla w\right] + \nabla p = f, \tag{2}$$

$$\mathrm{div}\, w = 0, \tag{3}$$

$$\frac{\partial k}{\partial t} + w\nabla k - \mathrm{div}\left[\left(\nu + c_\mu \frac{k^2}{\varepsilon}\right)\nabla k\right] - \frac{c_\mu}{2}\frac{k^2}{\varepsilon}\left|\nabla w + \nabla w^T\right|^2 + \varepsilon = 0,$$

$$\frac{\partial \varepsilon}{\partial t} + w\nabla \varepsilon - \mathrm{div}\left[\left(\nu + c_1 \frac{k^2}{\varepsilon}\right)\nabla \varepsilon\right] + c_3\frac{\varepsilon^2}{k} = c_2 k\left|\nabla w + \nabla w^T\right|^2. \tag{4}$$

The coefficients $c_\mu, c_1, c_2, c_3 \in \mathbb{R}$ are experimentally determined. The reader interested in a detailed description of the model is referred to [15].

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There is no result granting the existence of a solution to system (2)–(4) together with suitable boundary conditions. In fact, there are only some partial existence results isolating the difficulties associated to these equations.

In [10] and [11], the authors studied an elliptic equation which fits (4), assuming that both w and k are known, and with a boundary Dirichlet condition of the form $\varepsilon = \bar{\varepsilon}$. There are two remarkable difficulties arising in this problem. Firstly, since we consider $w \in H^1(\Omega)^N$ and $k \in L^\infty(\Omega)$ as input data, the term $k\left|\nabla w + \nabla w^T\right|^2$ in (4) belongs to $L^1(\Omega)$. Secondly, taking $u_2 = \varepsilon - \bar{\varepsilon}$ to obtain a homogeneous Dirichlet boundary condition, it can be checked that the diffusion coefficient blows up when $u_2 = -\bar{\varepsilon}$. Consequently, the term $\mathcal{A}_2(u_2)\nabla u_2$ is non-determined on this set and the search for weak solutions is not well-suited in this context. For this reason we use the framework of renormalized solutions. A renormalized solution for a parabolic problem like (4) was also discussed in [12].

This notion was first introduced by DiPerna and Lions, in [8], in the context of the Fokker–Planck–Boltzmann equation. Later on, it has been profusely used by many others authors for linear and nonlinear elliptic or parabolic problems (see, for example, [5,7,2] or [1]). It has also been applied to the study of nonlinear elliptic problems when the diffusion coefficient has a singularity for a finite value of the unknown [3,4].

In the present work, we analyze the coupling of the steady-state equation (4) with Eq. (2). Actually, we must face new difficulties. Another singular coefficient may appear in (2) (see Ref. [13]) though we do not consider here that situation. But, even more important, when dealing with the pressure term it is necessary to establish $\mathcal{A}_1, \mathcal{A}_3 \in L^\infty(\Omega)^N$ (see also [13]) in order to avoid a mixed variational formulation. Then we prefer to consider system (1) which should be understood as an intermediate problem arising in the analysis of the whole k–ε turbulence model. In these equations, the pressure term and the continuity equation are both dropped but some references involving this kind of argument can be found in [9].

This paper is organized as follows. Firstly, we give some known results which are used in this work. In Section 3, we enumerate the assumptions on data. In Section 4, we introduce the notion of a (renormalized) solution of system $(P1)$ and finally in Section 5, the existence result is established.

2. Some preliminary results

System (1) describes the coupling between an N-dimensional equation and a scalar one. Actually, the right-hand side in the second equation must be understood as an inner product in $\mathbb{R}^{N\times N}$ which is defined as follows

$$\forall A, B \in \mathbb{R}^{N\times N}, \qquad A : B = \mathrm{tr}(B^t A),$$

and the meaning of $\mathcal{A}_3(u_2)\nabla u_1 \nabla u_1$ is $\mathcal{A}_3(u_2)\nabla u_1 : \nabla u_1$.

Besides, $\langle\cdot,\cdot\rangle$ stands for the duality product between $H^{-1}(\Omega)^N$ and $H_0^1(\Omega)^N$ functions. Finally, $\mathcal{A}_i^{S/2}$ is the unique symmetric positive definite square root of matrix $\mathcal{A}_i^S$, the symmetric part of $\mathcal{A}_i$.

Throughout this paper, for every $j > 0$, $T_j(s)$ will denote the truncation function at height j, that is $T_j(s) = \operatorname{sgn} s \min(j, |s|)$, whereas sgn is the sign function. We also introduce the function $G_j(s)$ as follows

$$G_j(s) = T_{j+1}(s) - T_j(s) = \begin{cases} 0, & \text{if } |s| < j, \\ \operatorname{sgn} s, & \text{if } |s| \geq j+1, \\ s - j \operatorname{sgn} s, & \text{if } j \leq |s| \leq j+1. \end{cases} \tag{5}$$

The following lemma, due to Boccardo and Gallouët [6], will be used in the following; it is a very useful result in nonlinear elliptic equations with a right-hand side in $L^1(\Omega)$.

Lemma 1. *Let $(u_\delta)_\delta$ be a family of measurable functions such that*

(i) *$T_j(u_\delta) \in H_0^1(\Omega)$, for all $j > 0$, and*
(ii) *$\forall\delta > 0$ and $j > 0$, $\exists C > 0$ (independent of j and δ) such that $\int_\Omega |\nabla G_j(u_\delta)|^2 \leq C$.*

Then $(u_\delta)_\delta$ is bounded in $W_0^{1,q}(\Omega)$, for all q in the range $1 \leq q < N/(N-1)$.

Finally, we introduce the spaces

$$W_c^{1,\infty}(\mathbb{R}) = \left\{\varphi \in W^{1,\infty}(\mathbb{R}),\ \operatorname{supp}\varphi \text{ is compact}\right\}.$$

$$V = \{\phi \in H_0^1(\Omega)^N\ \mathcal{A}_1^{S/2}(u_2)\nabla\phi \in L^2(\Omega)^{N\times N}\}.$$

Functional spaces such as V, involving both data and the unknowns have been introduced in previous works. For instance, in [9], it appears in a similar framework. In fact, for the solution u_1 one may expect that $u_1 \in V$.