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Constructing an atlas for the diffeomorphism group of a compact manifold with boundary, with application to the analysis of image registrations

Carole J. Twining[a], Stephen Marsland[b,*]

[a] *Imaging Science and Biomedical Engineering (ISBE), University of Manchester, Manchester, UK*
[b] *Institute of Information Sciences, Massey University, Palmerston North, New Zealand*

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Abstract

This paper considers the problem of defining a parameterization (chart) on the group of diffeomorphisms with compact support, motivated primarily by a problem in image registration, where diffeomorphic warps are used to align images. Constructing a chart on the diffeomorphism group will enable the quantitative analysis of these warps to discover the normal and abnormal variation of structures in a population.

We construct a chart for particular choices of boundary conditions on the space on which the diffeomorphism acts, and for a particular class of metrics on the diffeomorphism group, which define a class of diffeomorphic interpolating splines. The geodesic equation is computed for this class of metrics, and we show how it can be solved in the spline representation. Furthermore, we demonstrate that the spline representation generates submanifolds of the diffeomorphism group, and we study this mapping. Explicit computational examples are included, showing how this chart can be constructed in practice, and that the use of the geodesic distance allows better classification of variation than those obtained using just a Euclidean metric on the space of warps.

MSC: 58D05; 53C22; 62H35

Keywords: Diffeomorphism group; Geodesics; Chart; Image registration

1. Introduction and overview

There has been much interest in the group of volume-preserving diffeomorphisms since Arnold's celebrated discovery [2] that the Euler fluid equations describe geodesics on the group of volume-preserving (i.e. incompressible) diffeomorphisms. This paper considers the construction of local charts on the group of all compactly-supported diffeomorphisms and the chart extension to form a coordinate system or atlas. We will show that the Euler equations for diffeomorphisms arise naturally from the construction of a right-invariant Riemannian metric on the diffeomorphism group. These Euler equations appear in several different fields, coinciding with the Camassa-Holm

* Corresponding address: Institute of Information Sciences & Technology, Massey University, Private Bag 11-222, Palmerston North, New Zealand. Tel.: +64 6 350 5799; fax: +64 6 350 2259.

E-mail addresses: carole.twining@manchester.ac.uk (C.J. Twining), S.R.Marsland@massey.ac.nz (S. Marsland).

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wave equation in fluid dynamics for the H^1 metric [13] and being used in two and three dimensions in the field of image registration.

It is this second application that provides the principal motivation for our work. We provide computational examples of how our methods can be used in Section 6, where we show that elements of the diffeomorphism group can be approximated to arbitrary accuracy on pixellated images based on a relatively small number of knotpoints, and Section 7 where we demonstrate that the relevant metric enables legal and illegal examples of 2D images of ventricles to be reliably classified.

Image warping is concerned with applying non-linear warps to an image in order to make structures in the image line up with corresponding structures in another (reference) image. There are a variety of applications, but the primary one is in medical imaging, where it is hoped that aligning images will enable the automated diagnosis of disease through analysis of the resultant deformation field. Under the assumption that registration between medical images should define a bijective, continuous, and invertible mapping between all points in the images, suitable warps come from some diffeomorphism group. Furthermore, in medical imaging the structures being warped are discrete, bounded entities. This suggests that the image warps should also be discrete and bounded, and so we restrict ourselves to some group of diffeomorphisms with compact support, $\mathcal{G} = \mathrm{Diff}_0(\mathcal{M})$. While there are places where such modelling choices are unsuitable (for example, where additional structure such as a tumour is seen in just one of the images) there are many other problems where they appear to be valid, such as degenerative brain diseases like Alzheimer's. For a general review of registration methods, and other applications, see [31], and for an overview of medical image registration, see [28].

Much of the most relevant work to this paper comes under the rubric of Computational Anatomy (see [23] for an overview), where the problem of inexact matching for landmarks, shapes and images is viewed as constructing orbits from a template under diffeomorphic transformations. Following [9,29], the problem of finding the minimum-distance image deformation under inexact matching is shown to have the same solution as the Euler–Lagrange equations, with methods such as geodesic shooting [22] used to construct an approximation to the true diffeomorphism. This differs from our approach where we consider only exact matching. In terms of our target application of analysing image deformation fields, this means that we analyse the exact field, not some approximation to it. While it may be that in the presence of noise the inexact matching provides a smoother deformation field, the derivation of the theory is far clearer when exact matching is considered. We also derive and consider the metric on the whole of $\mathrm{Diff}_0(\mathcal{M})$, rather than the metric on sets of landmarks (which is induced by the full metric) – see Section 5 for further details.

Further, this paper considers solving for geodesics on the diffeomorphism group within a spline representation, which is subtly but significantly different to the template matching approach of solving for geodesics on the space of knotpoint parameters using the induced metric [30,11,23].

The secondary purpose of our paper is to introduce the mathematics of the diffeomorphism group to those whose expertise is in the *application* of the methods in fields such as image registration. We therefore assume in the reader only knowledge of some standard differential geometry and functional calculus.

1.1. Problem statement

We consider the diffeomorphism group $\mathcal{G} = \mathrm{Diff}^s_0(\mathcal{M})$ of compactly-supported Sobolev H^s_0-mappings,[1] with identity element e. This group is a smooth (C^∞) Hilbert space, but it is *not* a Lie group [25,10]. In general, while these spaces are not Lie groups (group operations are not necessarily smooth between the same spaces except in the limit as s tends to infinity), the diffeomorphism groups do have structures corresponding to the Lie algebras and group exponential maps of Lie groups (see Section 2). This group exponential map is only continuous; there is no guarantee that any diffeomorphism in a neighbourhood of the identity can be embedded into a flow of vector fields [15,10].

The group acts on a space $\mathcal{M}$, of dimension $n < 2s$, which we will take to be a compact space with boundary $\partial\mathcal{M}$. Points on the boundary $\partial\mathcal{M}$ do not move, since the diffeomorphisms have compact support. We take $\mathcal{M}$ to be the closed unit ball $\overline{B}$ in $\mathbb{R}^n$. This simplifies some of the computations considerably, and fits well with the target application – the study of diffeomorphic warps of images – as the image plane/volume can be scaled to lie wholly within the unit ball.

[1] $H^s_0(\mathcal{M})$ is the space of functions f which vanish on the boundary $\partial\mathcal{M}$ (the meaning of the 0 subscript), and whose derivatives up to order s are square-integrable functions on $\mathcal{M}$.