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Spectral properties of primal-based penalty preconditioners for saddle point problems☆

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ABSTRACT

For large and sparse saddle point linear systems, this paper gives further spectral properties of the primal-based penalty preconditioners introduced in [C.R. Dohrmann, R.B. Lehoucq, A primal-based penalty preconditioner for elliptic saddle point systems, SIAM J. Numer. Anal. 44 (2006) 270–282]. The regions containing the real and non-real eigenvalues of the preconditioned matrix are obtained. The model of the Stokes problem is supplemented to illustrate the theoretical results and to test the quality of the primal-based penalty preconditioner.

1. Introduction

We are given the large, sparse and nonsingular linear system in saddle point form:

$$\mathcal{A}x \equiv \begin{pmatrix} A & B^{\mathrm{T}} \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ with $m \leq n$ (possibly $m \ll n$) and $C \in \mathbb{C}^{m\times m}$ symmetric positive semidefinite, which can arise, for example, in finite element discretizations of PDEs, see, e.g., [1–4], the geophysical inverse problems, see, e.g., [5], optimization problems, see, e.g., [6], and least-squares problems, see, e.g., [7]. We refer to Benzi, Golub, Liesen [8] for more applications and numerical solution techniques of (1).

This paper is devoted to making a further study on the preconditioner:

$$\mathcal{M} = \begin{pmatrix} I & B^{\mathrm{T}}\hat{C}^{-1} \\ 0 & -I \end{pmatrix} \begin{pmatrix} \hat{S} & 0 \\ 0 & -\hat{C} \end{pmatrix} \begin{pmatrix} I & 0 \\ \hat{C}^{-1}B & -I \end{pmatrix},$$

where $\hat{C}, \hat{S}$ are symmetric positive definite, and $\hat{S}$ is an approximation of the so-called inexact Schur complement $S = A + B^{\mathrm{T}}\hat{C}^{-1}B$. This preconditioner, called a primal-based penalty preconditioner, was recently presented by Dohrmann and Lehoucq [9] for elliptic saddle point systems. In the implementation of the preconditioned iteration, one application of $\mathcal{M}$ requires one application of $\hat{S}$ and two applications of $\hat{C}$. Assume that

$$\mathcal{H} = \begin{pmatrix} S - \hat{S} & 0 \\ 0 & \hat{C} - C \end{pmatrix}$$

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is symmetric positive definite. Then the bilinear form induced by $\mathcal{H}$

$$\langle x, y\rangle_{\mathcal{H}} := x^{\mathrm{T}}\mathcal{H}y$$

is an inner product. This inner product is called the $\mathcal{H}$-inner product. It is shown in [9] that $\mathcal{M}^{-1}\mathcal{A}$ is symmetric positive definite in the $\mathcal{H}$-inner product, i.e., $\mathcal{H}\mathcal{M}^{-1}\mathcal{A}$ is symmetric positive definite. Thus, the preconditioned Conjugate Gradient method which has short recurrences can be used. For the saddle point problem (1) with $C = 0$, when $\hat{C} = \varepsilon I$, $\hat{S} = A+\varepsilon^{-1}B^{\mathrm{T}}B$, the preconditioner $\mathcal{M}$ reduces to the regularized preconditioner which is first introduced by Axelsson [10], and then studied by Niet et al. [11,12] in detail. Recently, when $\hat{C} = \varepsilon M$, $S = A+B^{\mathrm{T}}\hat{C}^{-1}B$, the preconditioner $\mathcal{M}$ has been successfully applied to mixed finite elements approximations to the Stokes and Navier–Stokes equations in Gartling and Dohrmann [2]. Here, M is the pressure mass matrix. But no deep analysis was given. It should be mentioned that symmetric indefinite preconditioners similar to the primal-based penalty preconditioner has been deeply studied in [13,14,6,15].

This paper is to give a further analysis on the preconditioner $\mathcal{M}$. On the one hand, for the case $C = 0$, the estimates of the real and non-real eigenvalues of the preconditioned matrix are derived. On the other hand, the eigenvalue inclusion interval of $\mathcal{M}^{-1}\mathcal{A}$ is obtained under the mild conditions that $S - \hat{S}$ and $\hat{C} - C$ are symmetric positive definite. The interval obtained here only depends on three extreme values corresponding to $\hat{S}^{-1}S$ and $\hat{C}^{-1}B\hat{S}^{-1}B^{\mathrm{T}}$, whereas the bounds given in [9, Theorem 3.3] depend on five extreme values.

The remainder of this paper is organized as follows. In Section 2, we give regions containing the real and non-real eigenvalues of the preconditioned matrix $\mathcal{M}^{-1}\mathcal{A}$. In Section 3, we present numerical experiments to explain the theoretical results. Some concluding remarks are given in Section 4.

The following notations will be used throughout the paper. We denote by $\mathrm{Re}(\theta)$, $\mathrm{Im}(\theta)$, P^{T}, P^*, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ the real and the imaginary parts of the number θ, the transpose and the conjugate transpose of the matrix P, the minimum and the maximum real eigenvalues of the real matrix Q having real eigenvalues, respectively. We write $P \succ Q$ ($P \succeq Q$) if and only if $P - Q$ is symmetric positive definite (semidefinite) for real symmetric matrices P and Q.

2. Analysis of the preconditioner $\mathcal{M}$

In the implementation of the preconditioned iteration, the solution of the system

$$\mathcal{M}\begin{pmatrix} z_u \\ z_p \end{pmatrix} = \begin{pmatrix} r_f \\ r_g \end{pmatrix}$$

can reduce to

1. solve $\hat{S}z_u = r_f + B^{\mathrm{T}}\hat{C}^{-1}r_g$ for z_u;
2. solve $\hat{C}z_p = Bz_u - r_g$ for z_p.

Clearly, the preconditioner solves can be obtained by one solve with $\hat{S}$ and two solves with $\hat{C}$; see also [10,9]. When $\hat{C} = \varepsilon I$, $\hat{S} = A + \varepsilon^{-1}B^{\mathrm{T}}B$, it is obvious that the preconditioner solve can be derived only by one solve with $A + \varepsilon^{-1}B^{\mathrm{T}}B$.

Hereafter, we denote

$$\mu_1 = \lambda_{\min}(\hat{S}^{-1}S), \qquad \mu_n = \lambda_{\max}(\hat{S}^{-1}S), \qquad \tau_m = \lambda_{\max}(\hat{C}^{-1}C),$$
$$\nu_1 = \lambda_{\min}(\hat{C}^{-1}B\hat{S}^{-1}B^{\mathrm{T}}), \qquad \nu_m = \lambda_{\max}(\hat{C}^{-1}B\hat{S}^{-1}B^{\mathrm{T}}).$$

The following theorem gives the spectral properties of $\mathcal{M}$ for the saddle point problem (1) with $C = 0$.

Theorem 2.1. *Assume that $C = 0$. Then any eigenvalue λ of $\mathcal{M}^{-1}\mathcal{A}$ satisfies*

(i) *If* $\mathrm{Im}(\lambda) \neq 0$, *then* $|1-\lambda|^2 \leq 1 - \mu_1 < 1$ *and*

$$\mathrm{Re}(\lambda) \in [\chi_1, \chi_2], \mathrm{Im}(\lambda) \in [-\chi_3, \chi_3],$$

where

$$\chi_1 = \frac{1}{2}\mu_1, \qquad \chi_2 = \min\left\{1, \frac{1}{2}(\mu_n + \nu_m)\right\}, \qquad \chi_3 = \min\left\{\frac{1}{2}, \sqrt{\nu_m - \frac{\mu_1^2}{4}}\right\}.$$

(ii) *If* $\mathrm{Im}(\lambda) = 0$, *then*

$$\lambda \in [\chi_4, \chi_5],$$

where

$$\chi_4 = \begin{cases} \min\left\{\mu_1, \frac{1}{2}\left(\mu_n + \nu_1 - \sqrt{(\mu_n+\nu_1)^2 - 4\nu_1}\right)\right\} & \text{if } (\mu_n+\nu_1)^2 - 4\nu_1 \geq 0, \\ \mu_1 \quad \text{if } (\mu_n+\nu_1)^2 - 4\nu_1 < 0, \end{cases}$$
$$\chi_5 = \mu_n + \nu_m.$$