Journal of Computational and Applied Mathematics 233 (2010) 2285–2301

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Cubic superconvergent finite volume element method for one-dimensional elliptic and parabolic equations

Guanghua Gao *, Tongke Wang

School of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, PR China

ARTICLE INFO

Article history:
Received 21 May 2009
Received in revised form 30 August 2009

Keywords:
Elliptic equation
Parabolic equation
Cubic interpolation
Finite volume element method
Superconvergence
Optimal stress point

ABSTRACT

In this paper, a cubic superconvergent finite volume element method based on optimal stress points is presented for one-dimensional elliptic and parabolic equations. For elliptic problem, it is proved that the method has optimal third order accuracy with respect to H^1 norm and fourth order accuracy with respect to L^2 norm. We also obtain that the scheme has fourth order superconvergence for derivatives at optimal stress points. For parabolic problem, the scheme is given and error estimate is obtained with respect to L^2 norm. Finally, numerical examples are provided to show the effectiveness of the method.

1. Introduction

Finite volume element methods (FVEMs) [1–3], which are also called Petrov–Galerkin finite element methods, are the special cases of generalized difference methods (GDMs) [3–6]. The methods discretize the integral form of conservation law of differential equation by choosing linear or bilinear finite element space as trial space. They have the simplicity of finite difference methods and the accuracy of finite element methods and have been widely used in computational fluid mechanics because they keep the conservation law of mass or energy. In recent years, some literatures discussed finite volume element methods from different points of view. Cai and Steve Mccormick [1] presented finite volume element method for diffusion equations on composite grids and provided the error estimates which were relatively complicated. Afterwards, they gave simple theoretical analysis for diffusion equations on general triangulations. However, it was constrained to special choosing of control volumes. Li Qian and his colleagues [7,8] also had lots of contributions to the studies of finite volume element methods.

The development of efficient higher order finite volume methods is important both in theories and for various applications. Plexousakis and Zouraris [9] derived a class of high order finite volume methods for solving one-dimensional elliptic equations. Cai, Douglas and Park [10] constructed a high order finite volume element method by mixed variational principle. They presented a systematic way to derive high order finite volume method over rectangular meshes. Yang [11] constructed and analyzed a second order finite volume element scheme on general quadrilateral meshes for two-dimensional elliptic problem. In [12], Yang, Liu and Chen further developed the second order finite volume scheme with affine quadratic bases on three-dimensional right quadrangular prism grids for elliptic boundary problems. More work can be seen in References [13–23]. Xu and Zou [17] developed an abstract framework to give a unified presentation of finite volume methods and a unified study of the convergence theory of finite volume methods. Shu, Yu and Huang [20] presented a symmetric finite volume element scheme on quadrilateral grids and Wang [21] presented an alternating direction finite volume element method by perturbing the differential equations. By modifying trial function space, Wang [22,23] proposed some kinds of

* Corresponding author. Tel.: +86 15298362608.
E-mail addresses: gaoguanghua1107@163.com (G. Gao), wangtke@sina.com (T. Wang).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.10.013

high order finite volume schemes for one-dimensional elliptic and parabolic differential equations. Also many alternative finite volume schemes, e.g. WENO [24], ENO and FVs with embedded analytical functions [25] are constructed to obtain high order schemes.

Essentially, both finite element and finite volume element are methods based on interpolations. By approximation theory, we know that the numerical derivatives have only rth order accuracy for rth order interpolating polynomials in general. But this fact does not exclude the possibility of higher order accuracy of approximation for derivatives at some special points, which are called optimal stress points. By now the superconvergence theory of finite elements has clarified the distribution of the interpolation optimal stress points for some most in use finite elements [26,27]. For finite volume element methods, Li, Chen and Wu [3] gave the analysis with H^1 norm and L^2 norm for linear, quadratic and cubic Hermite finite volume element methods and discussed the superconvergence of linear and cubic element difference schemes. We know, for elliptic problem, the key step of finite volume element method is to discretize the normal derivatives of unknown function along the boundaries of control volumes. If the approximation order of these derivatives is higher, finite volume element methods may get higher accuracy. Guo and Wang [28] presented a high accuracy finite volume element method based on quadratic optimal stress points for two-point boundary value problem. In this paper, we will construct cubic superconvergent finite volume element method based on cubic optimal stress points. Our studies are motivated by the importance to obtain superconvergent finite volume element schemes, which can keep the local conversation, high order accuracy and get the superconvergence for derivatives at optimal stress points.

Let $\Pi_h u$ be the interpolating function over interval $[-h, h]$ associated with four equidistant nodes $(-h, u(-h))$, $(-h/3, u(-h/3))$, $(h/3, u(h/3))$, $(h, u(h))$, then

$$\Pi_h u = -\frac{(9\xi^2-1)(\xi-1)}{16}u(-h) + \frac{9(3\xi-1)(\xi^2-1)}{16}u\left(-\frac{h}{3}\right) - \frac{9(3\xi+1)(\xi^2-1)}{16}u\left(\frac{h}{3}\right) + \frac{(9\xi^2-1)(\xi+1)}{16}u(h),$$

where $\xi = x/h$. By Taylor's expansion, we have

$$(\Pi_h u)'(x_0) = u'(x_0) + \frac{1}{54}(5h^2x_0 - 9x_0^3)u^{(4)}(x_0) - \frac{1}{1080}(h^4 + 70h^2x_0^2 - 135x_0^4)u^{(5)}(x_0) + O(h^5).$$

Hence, when $x_0 = \pm\sqrt{5}h/3$ or $x_0 = 0$, $(\Pi_h u)'(x_0) = u'(x_0) + O(h^4)$. Therefore, the points $\{\pm\frac{\sqrt{5}}{3}, 0\}$ are optimal stress points of cubic Lagrange interpolation on the reference element $[-1, 1]$ when the interpolating nodes are equally distributed.

The remainder of paper is organized as follows. In Section 2, elliptic problem is discussed and error estimate is given. Superconvergence for derivatives is also included. In the following section, we extend the above ideas to one-dimensional parabolic problem. Finally, in Section 4, numerical examples are provided to show the effectiveness and adaption of the method.

Throughout this paper, we use C to denote a generic positive constant independent of discretization parameters.

2. Cubic superconvergent finite volume element method for one-dimensional elliptic equation

Consider the following one-dimensional elliptic equation with boundary values of mixed type on $I = [a, b]$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u(x) = f(x), \quad x \in (a, b), \tag{2.1a}$$

$$u(a) = 0, \qquad p(b)\frac{\mathrm{d}u(b)}{\mathrm{d}x} + \alpha u(b) = g, \tag{2.1b}$$

where p, q, f are given functions on $[a, b]$, and $p \in C^1[a, b]$, $q, f \in C[a, b]$, $p(x) \geq p_0 > 0$, $q(x) \geq 0$, $\alpha \geq 0$, g are constants. First, give a partition I_h on $I = [a, b]$ and the elements are $[x_{3(i-1)}, x_{3i}]$ $(i = 1, 2, \ldots, n)$. Then we divide each element into three equal parts with step size h_i and the nodes are $x_{3i-3} < x_{3i-2} < x_{3i-1} < x_{3i}$. Denote by $h = \max_{1\leq i\leq n} h_i$, $x_{3i-(3+\sqrt{5})/2} = x_{3i} - (3+\sqrt{5})h_i/2$, $x_{3i-3/2} = (x_{3i-2} + x_{3i-1})/2$, $x_{3i-(3-\sqrt{5})/2} = x_{3i} - (3-\sqrt{5})h_i/2$. From Section 1, we know that there are three optimal stress points of cubic interpolation on each element, which are $x_{3i-\frac{3+\sqrt{5}}{2}}$, $x_{3i-\frac{3}{2}}$, $x_{3i-\frac{3-\sqrt{5}}{2}}$. Let $I_i^* = \left[x_{3i-\frac{3+\sqrt{5}}{2}}, x_{3i-\frac{3}{2}}\right]$ $(i = 1, 2, \ldots, n)$, $I_i^{**} = \left[x_{3i-\frac{3}{2}}, x_{3i-\frac{3-\sqrt{5}}{2}}\right]$ $(i = 1, 2, \ldots, n)$, $I_i^{***} = \left[x_{3i-\frac{3-\sqrt{5}}{2}}, x_{3(i+1)-\frac{3+\sqrt{5}}{2}}\right]$ $(i = 1, 2, \ldots, n-1)$, $I_0^{***} = \left[x_0, x_{\frac{3-\sqrt{5}}{2}}\right]$, $I_n^{***} = \left[x_{3n-\frac{3-\sqrt{5}}{2}}, x_{3n}\right]$, then $\bigcup_i (I_i^* \bigcup I_i^{**} \bigcup I_i^{***})$ constructs the dual partition of I_h. All $I_i^*, I_i^{**}, I_i^{***}$ are also called control volumes. Integrating Eq. (2.1a) on I_i^*, I_i^{**}, I_i^{***} $(i = 1, 2, \ldots, n)$ and using integration formula by parts, we get the conservative integral form of (2.1), finding $u \in H_E^1(I) = \{u,\ u \in H^1(I),\ u(a) = 0\}$, such that

$$p_{3i-\frac{3+\sqrt{5}}{2}}u'\left(x_{3i-\frac{3+\sqrt{5}}{2}}\right) - p_{3i-\frac{3}{2}}u'\left(x_{3i-\frac{3}{2}}\right) + \int_{I_i^*} qu\,\mathrm{d}x = \int_{I_i^*} f\,\mathrm{d}x, \quad i = 1, 2, \ldots, n,$$