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Analysis of random walks with an absorbing barrier and chemical rule

Ahmed M.K. Tarabia *

Department of Mathematics, Damietta Faculty of Science, New Damietta 34517, Egypt

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ABSTRACT

Recently Tarabia and El-Baz [A.M.K. Tarabia, A.H. El-Baz, Transient solution of a random walk with chemical rule, Physica A 382 (2007) 430–438] have obtained the transient distribution for an infinite random walk moving on the integers $-\infty < k < \infty$ of the real line. In this paper, a similar technique is used to derive new elegant explicit expressions for the first passage time and the transient state distributions of a semi-infinite random walk having "chemical" rule and in the presence of an absorbing barrier at state zero. The walker starting initially at any arbitrary positive integer position i, $i > 0$. In random walk terminology, the busy period concerns the first passage time to zero. This relation of these walks to queuing problems is pointed out and the distributions of the queue length in the system and the first passage time (busy period) are derived. As special cases of our result, the Conolly et al. [B.W. Conolly, P.R. Parthasarathy, S. Dharmaraja, A chemical queue, Math. Sci. 22 (1997) 83–91] solution and the probability density function (PDF) of the busy period for the $M/M/1/\infty$ queue are easily obtained. Finally, numerical values are given to illustrate the efficiency and effectiveness of the proposed approach.

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1. Introduction

The difficulty of obtaining time-dependent solutions to continuous random walks is well known. In general, such problems tend to be intractable or provide solutions so complicated as to be of little practical use. A technique expounded in [6,7] and later in [11,12] is known as the series approach or randomization. This approach has proved successful in modeling several queueing situations and random walks. It is based on converting the system of differential-difference equations for the probabilities into a system of difference-equations using a series form. The coefficients of this series satisfy simple recurrence relations, which permit the rapid and efficient evaluation of the transition probabilities and the distribution of the first passage time to zero. Further simplifying this approach, we present a similar approach to obtain the exact distribution for the transition state solution and the first passage time distribution of the Conolly et al. [2] model. In such a model, a molecule can be modeled as an infinitely long chain of atoms joined by links of equal length. The links are subjected to random shocks and this causes the atoms to move and the molecule to diffuse. Moreover, we modify the actual process by making state 0 the absorbing state. This is a well studied model with considerable physical interest. For example, its physical applications with a similar probabilistic formalism arise in statistical mechanics in connection with the Ising model. In the apparently extremely simplified cases discussed in [3,9] and for other more applications see also [1]. Based on the generating function and Laplace transform technique, Conolly et al. [2] studied the transient distribution of a random walk and the first passage time T to reach zero. They also obtained the probability generating function $K(x, t)$ of the joint distributions of the length of a busy period and the number of customers served during that period in the chemical queue

* Corresponding address: Mathematics Department, Faculty of Science, El-Jouf University, P.O. 2014, Sakaka, Saudi Arabia. Tel.: +966 051159416; fax: +966 46255432.
E-mail addresses: a_tarabia@hotmail.com, a_tarabia@yahoo.com.

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as follows:

$$K(x,t)=\frac{abx}{a+b}\mathrm{e}^{-at}\left[1+\frac{1}{bt\sqrt{x}}I_1\left(at\sqrt{x}\right)+\frac{1}{at\sqrt{x}}I_1\left(bt\sqrt{x}\right)-\int_0^t\int_0^{t-s}\frac{I_1\left(au\sqrt{x}\right)}{u}\cdot\frac{I_1\left(b\left(t-s-u\right)\sqrt{x}\right)}{(t-s-u)}\mathrm{d}u\mathrm{d}s\right], \tag{1}$$

where $a=\lambda+\mu$, $b=\lambda-\mu$ and $I_n(.)$ is the modified Bessel function. The coefficient $k_n(t)$ of x^n in (1) is the joint probability of the length of a busy period and the number of customers served. It can be obtained in the form:

$$k_n(t)=\lambda\mu^2\mathrm{e}^{-at}\frac{t^{2n-2}}{(2n-2)!}d_n,\quad n\geq 2 \tag{2}$$

with

$$k_1(t)=\lambda\mathrm{e}^{-at},\qquad d_n=\sum_{m=0}^{n-2}d_{nm}\mu^{2m}\lambda^{2n-2m-4},$$

and

$$d_{nm}=\frac{1}{n-1}\binom{n-1}{m}\binom{n-1}{m+1},\quad 0\leq m\leq n-2.$$

In this paper, the motivation has been to give new and explicit expressions for the transient probabilities and the first passage time distribution of the random walks where the walker started initially at arbitrary state i, $i\geq 1$. Thus, we will not only give a general case of Conolly's et al. [2] model, but also to give new and simple formulas for their model.

This paper is organized as follows. Section 2 describes the model and the notation used. Then, we consider a direct approach to obtain the solutions of the Conolly et al. [2] model in elegant explicit formulas. In Section 3, we will show that the Conolly et al. [2] solution is a special case of our result. The derived formula for the transient state is free of Bessel functions or any integral forms. Moreover, we will demonstrate that the busy period distribution of a non-empty $M/M/1/\infty$ queueing model is a special case of our results. Finally, in Section 4 we give a brief discussion based on numerical calculation to illustrate the derived theoretical results.

2. The model and the main results

The general process considered in this section can be described as follows: A molecule is modeled as an infinitely long chain of atoms joined by links of equal length. The links are subjected to random shocks and this causes the atoms to move and the molecule to diffuse. In the more interesting case considered here the atoms are of two alternating kinds and the shock mechanism is different according to whether the atom occupies an odd or an even position on the chain. In additional, we assume the random walks have an absorbing barrier at state zero. It is easy to see that the given process can be described by a random walk $\{Q(t),t\geq 0\}$ with state space $E=\{0,1,2,3,\ldots\}$ where $Q(t)$ denotes the atom position on the real line up to time t, with $Q(0)=i$, $i\geq 1$. Its transition probabilities satisfy the following 'chemical rule' for any positive integers n and m, $n,m>0$: we have

$$\mathrm{prob}(Q(t+h)=m\mid Q(t)=n)=\begin{cases}\lambda h+o(h), & m=2k+1,n=2k\\ \mu h+o(h), & m=2k-1,n=2k\\ \lambda h+o(h), & m=2k,n=2k+1\\ \mu h+o(h), & m=2k,n=2k-1,\end{cases}$$

and

$$\mathrm{prob}(Q(t+h)=m\mid Q(t)=n)=\begin{cases}\lambda h+o(h), & m=0,n=1\\ 0, & m=1,n=0,\end{cases}$$

where λ and μ are positive constants. Also let

$$q_{i,n}(t)=\mathrm{prob}(Q(t)=n\mid Q(0)=i),\quad i,n=0,1,2\ldots$$

or simply $q_n(t)$. From the above assumptions, the Chapman–Kolmogorov forward equations for the system are

$$\dot{q}_0(\tau)=\rho\,q_1(\tau),\quad n=0, \tag{3}$$

$$\dot{q}_1(\tau)=-(1+\rho)\,q_1(\tau)+q_2(\tau),\quad n=1, \tag{4}$$

$$\dot{q}_{2l}(\tau)=-(1+\rho)\,q_{2l}(\tau)+q_{2l-1}(\tau)+\rho\,q_{2l+1}(\tau),\quad n=2l,\ l\geq 1 \tag{5}$$

$$\dot{q}_{2l+1}(\tau)=-(1+\rho)\,q_{2l+1}(\tau)+\rho\,q_{2l}(\tau)+q_{2l+2}(\tau),\quad n=2l+1,\ l\geq 1 \tag{6}$$

$$q_n(0)=\delta_{i,n},\quad i\geq 1 \tag{7}$$

with $\dot{q}_n(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}q_n(\tau)$, $\delta_{i,n}$ is the Kronecker's delta, $\rho=\lambda/\mu$ and the time has been scaled such that $\tau=\mu t$.

The following theorem without proof, called randomization or uniformization, is a well-known result used primarily for the numerical computation of the transition probability $q_n(t)$ of a Markov process (cf., [4]).

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