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Oscillatory criteria for Third-Order difference equation with impulses

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ABSTRACT

In this paper, we investigate the oscillation of Third-order difference equation with impulses. Some sufficient conditions for the oscillatory behavior of the solutions of Third-order impulsive difference equations are obtained.

1. Introduction

Consider the impulsive difference equation

$$\begin{cases} \triangle^3 x(n) + p(n)x(n-\tau) = 0, & n \neq n_k, k = 1, 2, 3\dots \\ x(n_k) = a_k x(n_k - 1), & k = 1, 2, 3\dots \\ \triangle x(n_k) = b_k \triangle x(n_k - 1), & k = 1, 2, 3\dots \\ \triangle^2 x(n_k) = c_k \triangle^2 x(n_k - 1), & k = 1, 2, 3\dots \end{cases} \tag{1}$$

where $a_k > 0, b_k > 0, c_k > 0, p(n) \geq 0, p(n) \not\equiv 0, 0 < n_0 < n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_{k\to\infty} n_k = \infty, \tau \in N, \triangle x(n) = x(n+1) - x(n)$.

It is well known that equations with impulses have been considered by many authors. The theory of impulsive differential/difference equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential/difference equations without impulse effects. Moreover, such equations may exhibit several real-world phenomena, such as rhythmical beating, merging of solutions, and noncontinuity of solutions.

In recent years, there has been increasing interest in the oscillation/nonoscillation of impulsive differential/difference equations, and numerous papers have been published on this class of equations and good results were obtained (see [1–8, 10–15] etc. and the references therein). But there are fewer papers on impulsive difference equations [5–7].

For example, in [5], Mingshu Peng researched the equation

$$\triangle(r_{n-1}|\triangle(x_{n-1} - x_{n-\tau-1})|^{\alpha-1}\triangle(x_{n-1} - x_{n-\tau-1})) + f(n, x_n, x_{n-l}) = 0,$$
$$r_{n_k}|\triangle(x_{n_k} - x_{n_k-\tau})|^{\alpha-1}\triangle(x_{n_k} - x_{n_k-\tau}) = M_k(r_{n_k-1}|\triangle(x_{n_k-1} - x_{n_k-\tau-1})|^{\alpha-1}\triangle(x_{n_k-1} - x_{n_k-\tau-1})).$$

He obtained sufficient conditions for oscillation of all solutions of the equation.

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In [3], Wu Xiu-Li et al. discussed the equation

$$[r(t)x'(t)]' + p(t)x'(t) + Q(t, x(t)) = 0, \quad t \geq t_0, t \neq t_k, k = 1, 2, \ldots,$$
$$x(t_k^+) = g_k(x(t_k)), \qquad x'(t_k^+) = h_k(x'(t_k)),$$
$$x(t_0^+) = x_0, \qquad x'(t_0^+) = x'_0.$$

They also investigated the oscillation of the above equation.

In [10], Wan and Mao paid attention to the following system

$$\begin{cases} x'''(t) + p(t)x(t) = 0, \quad t \geq t_0, t \neq t_k, k = 1, 2, 3 \ldots \\ x(t_k^+) = a_k x(t_k), \qquad x'(t_k^+) = b_k x'(t_k), \qquad x''(t_k^+) = c_k x''(t_k), \\ x(t_0^+) = x_0, \qquad x'(t_0^+) = x'_0, \qquad x''(t_0^+) = x''_0. \end{cases}$$

The sufficient conditions are obtained for all solutions either oscillating or asymptotically tending to zero.

In this paper, we study Eq. (1) and we get some sufficient conditions for the oscillation of solutions of Eq. (1).

Definition 1. By a solution of (1) we mean a real-valued sequence $\{x_n\}$ defined on $\{n_0 - \tau, n_0 - \tau + 1, n_0 - \tau + 2, \ldots\}$ which satisfies (1) for $n \geq n_0$.

Definition 2. A solution of Eq. (1) is said to be nonoscillatory if the solution is eventually positive or eventually negative; otherwise, the solution is said to be oscillatory.

This paper is organized as follows. In Section 2, we shall offer some lemmas and theorems. To illustrate our results, some examples are included in Section 3.

2. Main results

In order to prove our theorems, we need the following lemmas.

Lemma 1. *Assume that $x(n)$ is a solution of (1), and the following conditions are satisfied:*
H1*:* $(n_1 - n_0) + b_1(n_2 - n_1) + b_1b_2(n_3 - n_2) + \cdots + b_1b_2b_3\cdots b_m(n_{m+1} - n_m) + \cdots = \infty$,
H2*:* $(n_1 - n_0) + c_1(n_2 - n_1) + c_1c_2(n_3 - n_2) + \cdots + c_1c_2c_3\cdots c_m(n_{m+1} - n_m) + \cdots = \infty$, *for some* $i \in \{1, 2\}$, *there exists* $N \geq n_0$ *such that* $\triangle^{i+1} x(n) \geq 0 (\leq 0)$, $\triangle^i x(n) > 0 (< 0)$ *for* $n \geq N$. *Then* $\triangle^{i-1} x(n) > 0 (< 0)$ *holds for sufficiently large n.*

Proof. We only prove the conclusion under the assumption that $\triangle^{i+1} x(n) \geq 0$, $\triangle^i x(n) > 0$. Without loss of the generality, suppose $N = n_0$. From $\triangle^{i+1} x(n) \geq 0$, we know that $\triangle^i x(n)$ is monotonically nondecreasing in $[n_k, n_{k+1})$, $k = 0, 1, 2, \ldots$. Hence

$$\triangle^i x(n) \geq \triangle^i x(n_k), \quad n \in [n_k, n_{k+1}).$$

Summing the above inequality from n_k to $n_{k+1} - 1$, we have

$$\triangle^{i-1} x(n_{k+1}) \geq \triangle^{i-1} x(n_k) + \triangle^i x(n_k)(n_{k+1} - n_k). \tag{2}$$

So

$$\triangle^{i-1} x(n_2) \geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1),$$

thus

$$\begin{aligned} \triangle^{i-1} x(n_3) &\geq \triangle^{i-1} x(n_2) + \triangle^i x(n_2)(n_3 - n_2) \\ &\geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1) + d_2 \triangle^i x(n_2 - 1)(n_3 - n_2) \\ &\geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1) + d_2 \triangle^i x(n_1)(n_3 - n_2) \end{aligned}$$

where

$$d_k = \begin{cases} c_k, & i = 2, \\ b_k, & i = 1. \end{cases}$$

By induction, we get

$$\triangle^{i-1} x(n_k) \geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)[(n_2 - n_1) + d_2(n_3 - n_2) + \cdots + d_2d_3\cdots d_{k-1}(n_k - n_{k-1})].$$

From (H1) or (H2), we know that there exists l such that $\triangle^{i-1} x(n_k) > 0$ for $k \geq l$. Since $\triangle^i x(n) > 0$, we get

$$\triangle^{i-1} x(n) > \triangle^{i-1} x(n_k) > 0, \quad n \in [n_k, n_{k+1}), n_k \geq n_l.$$

We complete the proof. □

Lemma 2. *Let $x(n)$ be a solution of Eq. (1) and,* (H1) *and* (H2) *hold, for some* $i \in \{1, 2, 3\}$, *there exists a constant* $N (N \geq n_0)$, *such that* $x(n) > 0$, $\triangle^i x(n) \leq 0$, $\triangle^i x(n) \not\equiv 0$ *in any interval* $[n, \infty)$. *Then* $\triangle^{i-1} x(n) > 0$ *holds for sufficiently large n.*