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On quasi-consistent integration by Nordsieck methods ☆

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ABSTRACT

In this paper we study advantages of numerical integration by quasi-consistent Nordsieck formulas. All quasi-consistent numerical methods possess at least one important property for practical use, which has not attracted attention yet, i.e. the global error of a quasi-consistent method has the same order as its local error. This means that the usual local error control will produce a numerical solution for the prescribed accuracy requirement if the principal term of the local error dominates strongly over remaining terms. In other words, the global error control can be as cheap as the local error control in the methods under discussion.

Here, we apply the above-mentioned idea to Nordsieck–Adams–Moulton methods, which are known to be quasi-consistent. Moreover, some Nordsieck–Adams–Moulton methods are even super-quasi-consistent. The latter property means that their propagation matrices annihilate two leading terms in the defect expansion of such methods. In turn, this can impose a strong relation between the local and global errors of the numerical solution and allow the global error to be controlled effectively by a local error control. We also introduce Implicitly Extended Nordsieck methods such that in some sense they form pairs of embedded formulas with their source Nordsieck methods. This facilitates the local error control in quasi-consistent Nordsieck schemes. Numerical examples presented in this paper confirm clearly the power of quasi-consistent integration in practice.

1. Introduction

The notion of quasi-consistent numerical integration of ordinary differential equations (ODEs) of the form

$$x'(t) = g(t, x(t)), \quad t \in [t_0, t_{\mathrm{end}}], \qquad x(t_0) = x^0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $g : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a sufficiently smooth function, is new. It means merely an integration conducted by a quasi-consistent numerical method. Despite the fact that the property of quasi-consistency was discovered in [23] in 1976 it has not attracted researchers' attention, perhaps because of difficulties arising in practical implementation.

The first and most well-known quasi-consistent methods are Nordsieck–Adams–Moulton Formulas (NAMF). Skeel and Jackson [25] studied consistency and quasi-consistency of Nordsieck methods in 1977. Nordsieck methods themselves were introduced in 1962 (see [20]). In some sense they are equivalent to fixed-stepsize multistep formulas (see [24] or [8, p. 412–417]).

There are available a number of efficient software packages implementing this and variants of the Nordsieck representation of multistep formulas. For example, we mention here EPISODE (see [5,12]), DIFSUB (see [6]), GEAR (see [10]) and LSODE (see [11]). Unfortunately, even when implemented effectively, adaptive Nordsieck formulas cannot provide

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quasi-consistent integration (as defined above) because of the order reduction phenomenon, which appears on variable meshes (see [16]). This means that any adaptive Nordsieck scheme cannot enjoy the benefit of quasi-consistent integration when implemented on a non-equidistant grid in the usual way. Super-convergent explicit peer methods designed recently in [27] seem to be the only numerical technique that is able to provide quasi-consistent integration on variable meshes for the moment. They are a subclass of s-stage general linear methods. General linear methods are studied in detail in [4]. Notice that general linear methods, in general, as well as peer methods, in particular, admit the Nordsieck representation (see, for example, [2–4,21]).

In this paper we explore an advantage of quasi-consistent integration by NAMF methods. More precisely, we intend to use quasi-consistency for an efficient global error control. This property ensures that any quasi-consistent numerical method does not accumulate the principal term of its local error in the original form and multiply it by the stepsize before accumulation (see [23]). In other words, only higher order terms are accumulated and contribute to the global error in the course of numerical integration. So, if the accumulated error does not exceed the principal term of the local error (both values are of the same order with respect to the stepsize) then the principal term will give us a reasonable estimate of the true error. In turn, the principal term of the local error can be easily evaluated and controlled.

This idea is much more effective in doubly quasi-consistent methods introduced here. The property of double quasi-consistency implies that the methods possessing it accumulate errors of two order higher than the local error of the numerical solution. This means that the principal terms of the local and global errors of a doubly quasi-consistent method coincide. Therefore the principal term of the local error is an asymptotically correct estimate of the true error for any doubly quasi-consistent method when the stepsize is sufficiently small.

Unfortunately, we prove below that Nordsieck schemes cannot be doubly quasi-consistent. So the search for doubly quasi-consistent methods is to be done among general linear methods, which have more freedom in choosing coefficients. Nevertheless, Kulikov and Shindin [17] prove that all $(r+1)$-value Nordsieck formulas of order $r+1$ are super-quasi-consistent if r is an even integer, i.e their propagation matrices annihilate two leading terms in the defect expansion. This results in strong correlation of the local and global errors of such numerical schemes and allows the global error to be controlled via local error estimates. Thus, all NAMF methods of odd order are super-quasi-consistent and a good choice for checking the new idea of global error control in practice.

Once again, the property of quasi-consistency does not work in variable-stepsize Nordsieck methods. That is why only a fixed-stepsize implementation is considered in this paper. Certainly, it is not efficient and serves for illustration of our theory of global error control presented below. However, adaptivity can be incorporated into fixed-stepsize NAMF methods in the same way as is done in geometric integration methods, i.e. we do not change the stepsize but rather use a time transformation of the problem under solution (see, for example, Chapter VIII in [7]).

To facilitate the local error estimation technique presented in [17] for Nordsieck methods we exploit the concept of extended Nordsieck formulas. These new Nordsieck formulas intend to produce numerical solutions that are asymptotically equal, as the stepsize tends to zero, to those obtained in local extrapolation of numerical solutions of source Nordsieck formulas. In other words, the extended Nordsieck methods are more accurate than the original ones. The profit of such an extension is explained clearly at the end of Section 3.

The remainder of this paper is organized as follows: In Section 2 we give a brief description of Nordsieck methods on uniform grids. It is also supplied with precise definitions of quasi-consistency, essential-consistency, double quasi-consistency and double essential-consistency. Section 3 proves non-existence of doubly quasi-consistent Nordsieck schemes and describes the objective of this paper at large. Section 4 introduces extended Nordsieck formulas and studies their convergence. Such methods facilitate significantly the local (and global) error estimation in quasi-consistent Nordsieck formulas. An algorithm of stepsize selection is also discussed there. Section 5 contains numerical tests that confirm the power of quasi-consistent numerical integration, and the last section summarizes the results obtained in the paper and outlines future plans.

2. Quasi-consistent Nordsieck methods

In this section, we give definitions and formulate previously obtained results that are used below.

Having introduced an equidistant mesh

$$w_\tau = \{t_k = t_0 + k\tau, k = 0, 1, \dots, K, K\tau = t_{\text{end}} - t_0\}$$

with a stepsize τ on the integration segment $[t_0, t_{\text{end}}]$, we define:

Definition 1. A vector of the form

$$X(t) = \left(x(t), \tau x'(t), \frac{\tau^2}{2!}x''(t), \dots, \frac{\tau^r}{r!}x^{(r)}(t)\right)^{\mathrm{T}}$$

is referred to as a Nordsieck vector of dimension $r+1$ for a sufficiently smooth scalar function $x(t)$.