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The great theorem of A.A. Markoff and Jean Bernoulli sequences

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ABSTRACT

A proof of Markoff's Great Theorem on the Lagrange spectrum using continued fractions is sketched. Markoff's periods and Jean Bernoulli sequence[1] are used to obtain a simple algorithm for the computation of the Lagrange spectrum below 3.

1. Introduction

Lagrange wrote an important addendum [1] to Euler's Algebra [2] were he presented his theory of binary quadratic forms. Lagrange's approach was based on Euler's study of periodic continued fractions, which in turn was stimulated by Brouncker's answer to a question of Fermat on Pell's equation. In 1880 A.A. Markoff completed his Magister Thesis in St.Petersburg University devoted to the theory of binary quadratic forms of positive determinant. This was an outstanding research in which A.A. Markoff presented his great discoveries in the field. The results were published in [3,4] exactly as they were presented in the thesis. Later Markoff returned to this topic in [5]. Let

$$f(x, y) = ax^2 + bxy + cy^2, \qquad D(f) = b^2 - 4ac$$

be an indefinite quadratic form of the determinant $D(f)$ considered as a function on the lattice $S = (\mathbb{Z} \times \mathbb{Z}) \setminus \{(0, 0)\}$. Two quadratic forms f and f^{equiv} are called equivalent if there exist integers $m, n, k, l, ml - nk = \pm 1$, such that

$$f^{equiv}(mx + ny, kx + ly) = f(x, y).$$

If $D(f)$ is not a perfect square in $\mathbb{Z}$, then $\inf\{|f(x, y)| : (x, y) \in S\}$, which we denote by $\inf(f)$, is positive. Since equivalent quadratic forms have equal determinants, the following functional is invariant

$$\mathrm{Mar}(f) = \frac{\sqrt{D(f)}}{\inf(f)}$$

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1 Following Markoff [A.A. Markoff, Sur une question de Jean Bernoulli, Math. Ann. 19 (1882), 27–36] we assume that Johann(III) Bernoulli = Jean Bernoulli.

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under the following transformations $f \to \lambda f$ and $f \to f^{equiv}$. The range of $\mathrm{Mar}(f)$ is called the *Markoff spectrum*.

Reversing Lagrange's arguments in [1] to return back to Euler we consider regular continued fractions. The Lagrange function $\mu(\xi)$ is defined for irrational ξ as the supremum of $c > 0$ such that

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{cq^2}$$

has infinitely many solutions in integers $p, q, q > 0$. So, the greater the $\mu(\xi)$ is the better the ξ can be approximated by rational numbers. The range of μ is called the *Lagrange spectrum*. Both spectra coincide in [0, 3], see [6].

Markoff proved that the Lagrange spectrum is discrete in [0, 3] with the only accumulation point 3. On the one hand any ξ with $\mu(\xi) < 3$ is a quadratic irrationality which is equivalent to the continued fraction

$$\xi(\theta,\delta) = \frac{1}{r_1} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_3} + \cdots + \frac{1}{r_n} + \frac{1}{r_n} + \cdots, \tag{1}$$

with $\delta = 0$ and a rational θ, $1 \leqslant \theta \leqslant 2$. Here

$$r_n = [(n+1)\theta + \delta] - [n\theta + \delta]$$

is a Jean Bernoulli sequence which Jean Bernoulli introduced in his treaty on Astronomy [7]. On the other hand $\mu(\xi(\theta,\delta)) = 3$ for irrational $\theta \in (1, 2)$. Hence there are transcendental numbers with $\mu(\xi) = 3$. Moreover, they may be represented by regular continued fractions (1), which are simply expressed via Jean Bernoulli sequences. These transcendental numbers, as it follows from Markoff's main result, in contrast to Liouville's constant

$$L = \sum_{n=0}^{\infty} 10^{-n!} = \frac{1}{9} + \frac{1}{11} + \frac{1}{99} + \frac{1}{1} + \frac{1}{10} + \frac{1}{9} + \frac{1}{999999999999} + \frac{1}{1} + \cdots$$

are the worst transcendental numbers from the point of view of rational approximation. Markoff's results were revised several times and his proofs were considerably simplified by eliminating continued fractions, see [8]. However the price for such a simplification is that a very important relationship with Jean Bernoulli sequences just disappeared. The approach of [6] strongly modifies the original Markoff's argument since basically it is addressed to the study of the differences between two spectra above 3. In this paper we sketch an approach which clarifies Markoff's logic and his motivations. As an application we give a simple algorithm for calculating the Lagrange spectrum below 3.

2. Motivations for Markoff's result

The first values of the Lagrange spectrum can be calculated in an elementary way, see for instance [9]. These calculations as well as further Markoff's theory are based on the formulas going back to Euler and Lagrange

$$\left|\xi - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n^2} \cdot \frac{1}{\frac{Q_{n-1}}{Q_n} + \xi_{n+1}}, \qquad \frac{Q_{n-1}}{Q_n} = \frac{1}{b_n} + \cdots + \frac{1}{b_1}, \qquad \xi_{n+1} = b_{n+1} + \frac{1}{b_{n+2}} + \cdots \tag{2}$$

$$\mu(\xi) = \limsup_n \left\{ \left(\frac{1}{b_n} + \cdots + \frac{1}{b_1} \right) + b_{n+1} + \left(\frac{1}{b_{n+2}} + \frac{1}{b_{n+3}} + \cdots \right) \right\}. \tag{3}$$

We define by $m(r)$ the same quantity for sequences $r = \{r_n\}_{n\in\mathbb{Z}}$. It looks like Markoff did such calculations for at least ten first values to obtain the following table (it is given in an addendum to Markoff's thesis):

ξ	$\mu(\xi)$	
$[\overline{1}]$	$\sqrt{5}$	$= 2.236067977\ldots$
$[\overline{2}]$	$\sqrt{8}$	$= 2.828427125\ldots$
$[\overline{2_2, 1_2}]$	$\frac{\sqrt{221}}{5}$	$= 2.973213749\ldots$
$[\overline{2_2, 1_4}]$	$\frac{\sqrt{1517}}{13}$	$= 2.996052630\ldots$
$[\overline{2_4, 1_2}]$	$\frac{\sqrt{7565}}{29}$	$= 2.999207188\ldots$
$[\overline{2_2, 1_6}]$	$\frac{\sqrt{2600}}{17}$	$= 2.999423243\ldots$
$[\overline{2_2, 1_8}]$	$\frac{\sqrt{71285}}{89}$	$= 2.999915834\ldots$
$[\overline{2_6, 1_2}]$	$\frac{\sqrt{257045}}{169}$	$= 2.999976658\ldots$
$[\overline{2_2, 1_2, 2_2, 1_4}]$	$\frac{\sqrt{84680}}{97}$	$= 2.999982286\ldots$
$[\overline{2_2, 1_{10}}]$	$\frac{\sqrt{488597}}{233}$	$= 2.999987720\ldots$

The periods of the continued fractions given in the first column consist of repeating 1's and 2's. For any sequence $r = \{\ldots, r_1, r_2, r_3, \ldots\}$ let $r^* = \{\ldots, r_1, r_1, r_2, r_2, r_3, r_3, \ldots\}$. It is clear from Markoff's remarks that he knew that Jean Bernoulli