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Higher order hypergeometric Lauricella function and zero asymptotics of orthogonal polynomials

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ABSTRACT

The asymptotic contracted measure of zeros of a large class of orthogonal polynomials is explicitly given in the form of a Lauricella function. The polynomials are defined by means of a three-term recurrence relation whose coefficients may be unbounded but vary regularly and have a different behaviour for even and odd indices. Subclasses of systems of orthogonal polynomials having their contracted measure of zeros of regular, uniform, Wigner, Weyl, Karamata and hypergeometric types are explicitly identified. Some illustrative examples are given.

1. Introduction

It is an usual way of working in quantum many-body physics to transform the Hamiltonian operator of a physical system into an N-dimensional Jacobi matrix by means of the Lanczos algorithm or any of its numerous variants [1,2]. It is also known that for a general $N \times N$ Jacobi matrix the characteristic polynomials of the principal submatrices form a set of orthogonal polynomials $\{P_n(x)\}_{n=1}^{N}$ which satisfy the recurrence relation

$$\begin{aligned} &P_n(x) = (x - a_n)\, P_{n-1}(x) - b_n^2\, P_{n-2}(x), \quad n = 1, 2, \ldots \\ &P_{-1}(x) = 0, \qquad P_0(x) = 1, \end{aligned} \tag{1}$$

where a_n and b_n are the Jacobi entries. It happens that the zeros of these orthogonal polynomials denote the energies of the levels of the physical system [3]. Here, following [4,5], we consider the class of systems of orthogonal polynomials defined by a recurrence relation of the previous type with coefficients satisfying the asymptotic conditions

$$\lim_{n\to\infty} \frac{a_{2n}}{\lambda_{2n}} = \alpha_1, \qquad \lim_{n\to\infty} \frac{b_{2n}}{\lambda_{2n}} = \beta_1, \qquad \lim_{n\to\infty} \frac{a_{2n+1}}{\lambda_{2n}} = \alpha_2, \qquad \lim_{n\to\infty} \frac{b_{2n+1}}{\lambda_{2n}} = \beta_2, \tag{2}$$

where $\lambda_n = g(n)$ is a regular varying function with exponent $\alpha \geq 0$. A regular varying function with exponent α can be written [6] as $g(x) = x^{\alpha} L(x)$ where $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function which satisfies $\lim_{x\to\infty} L(xt)/L(x) = 1$.

Associated with orthogonal polynomials of this type, with bounded ($\lambda_n = 1$ or $\alpha = 0$ and $L(x) = 1$) and unbounded ($\alpha > 0$) coefficients there exist a great variety of physical systems [7,8,2,9].

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2. Background and notations

For each polynomial $P_n(x)$, as defined by the recurrence (1), we consider its contracted and normalized zero counting measure

$$\rho_n := \frac{1}{n}\sum_{j=1}^{n}\delta\left(\frac{x_{j,n}}{\lambda_n}\right)$$

where $x_{j,n}$, $(j = 1, \ldots, n)$ are the zeros of $P_n(x)$, $\delta(x_{j,n}/\lambda_n)$ denotes the Dirac point mass at the scaled zero $x_{j,n}/\lambda_n$ and the scaling factor λ_n is the nth element of the regular varying sequence such that the asymptotic behaviour shown in (2) holds true. It could be interesting to remark that when the family $\{P_n(x)\}_n$ satisfies a holonomic linear second order differential equation, the corresponding scaling factor can be obtained in terms of the coefficients characterizing such an equation (see [10] for details).

Our aim here is to express in terms of higher order hypergeometric Lauricella functions the corresponding asymptotic contracted measure of zeros for the sequence $\{P_n(x)\}_{n=1}^{N}$ to be denoted by ρ, i.e. a probability measure that satisfies

$$\lim_{n\to\infty}\int f\,\mathrm{d}\rho_n = \int f\,\mathrm{d}\rho$$

for every continuous function f on $\mathbb{R}$ that vanishes at ∞. For this purpose, let us first introduce the following parameters (to be used throughout the paper)

$$\beta = \left[\frac{1}{4}\,(\alpha_1-\alpha_2)^2 + (\beta_1+\beta_2)^2\right]^{1/2}, \qquad \gamma = \frac{1}{2}\,(\alpha_1+\alpha_2)\,,$$
$$\delta = \left[\frac{1}{4}\,(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\right]^{1/2}, \tag{3}$$

where α_i and β_i $(i = 1, 2)$ are the limits given in the above expressions (2). On doing this, the following well known result of van Assche [4] will play a essential role:

Proposition 1 (*Theorem 4 (iii), [4]*). *Let $\{P_n(x)\}_{n=1}^{\infty}$ be an orthogonal polynomial sequence satisfying the recurrence (1), such that the a_n and b_n coefficients behave asymptotically as in (2). Then,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(x_{j,n}/\lambda_n) = \int_0^1\int_{-\infty}^{+\infty} f(x)\mathrm{d}F(x-\gamma t^{\alpha};\delta t^{\alpha},\ \beta t^{\alpha})\mathrm{d}t, \tag{4}$$

for every continuous function f. Here:

(a) $\{\lambda_n\}_{n\in\mathbb{N}}$ *is a regularly varying sequence with exponent* $\alpha \geq 0$.
(b) β, γ *and* δ *are the numbers defined in* (3).
(c) $F(x; u, v) = \frac{1}{\pi}\int_{-\infty}^{x}\frac{|t|}{(v^2-t^2)^{1/2}(t^2-u^2)^{1/2}}\,\mathbb{I}_B(t)\mathrm{d}t$, *with* $B = [-v, -u]\bigcup[u, v]$ *being*

$$\mathbb{I}_B(t) = \begin{cases} 1 & \text{if } t \in B \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand, the F_D-Lauricella function of order n is defined by the series (cf. [11])

$$F_D^{(n)}\left[\begin{matrix} a\,;\, b_1\,,\,\ldots\,,\,b_n\,;\,c \\ x_1\,,\,\ldots\,,\,x_n \end{matrix}\right] = \sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a, m_1+\cdots+m_n)(b_1, m_1)\cdots(b_n, m_n)\,x_1^{m_1}\cdots x_n^{m_n}}{(c, m_1+\cdots+m_n)m_1!\ldots m_n!} \tag{5}$$

where $(a, j) = (a)_j$ stands for the Pochhammer symbol. Among others, this function satisfies the following reduction properties:

(a) If two variables coincide (e.g. $x_i = x_{i+1}$) then

$$F_D^{(n)}\left[\begin{matrix} a\,;\, b_1\,,\,\ldots\,,\,b_{i-1}\,,\,b_i\,,\,b_{i+1}\,,\,\ldots\,,\,b_n\,;\,c \\ x_1\,,\,\ldots\,,\,x_{i-1}\,,\,x_i\,,\;\;x_i\;\;,\,\ldots\,,\,x_n \end{matrix}\right] = F_D^{(n-1)}\left[\begin{matrix} a\,;\, b_1\,,\,\ldots\,,\,b_{i-1}\,,\,b_i+b_{i+1}\,,\,\ldots\,,\,b_n\,;\,c \\ x_1\,,\,\ldots\,,\,x_{i-1}\,,\qquad x_i\qquad,\,\ldots\,,\,x_n \end{matrix}\right]. \tag{6}$$

(b) In particular, the Lauricella function of two arguments ($F_D^{(2)}$) reduces to the so-called Appell hypergeometric function F_1:

$$F_D^{(2)}\left[\begin{matrix} a\,;\, b_1\,,\,b_2\,;\,c \\ x_1\,,\,x_2 \end{matrix}\right] = F_1[a, b_1, b_2; c; x_1, x_2], \tag{7}$$

where the notation of [12] for F_1 is used.

With this background at hand, we are now in a position to show how the Lauricella $F_D^{(5)}$ function appears when trying to obtain the aforementioned asymptotic contracted measure of the zeros.