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An algorithm for semi-infinite transportation problems

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Abstract

In this paper we consider a class of semi-infinite transportation problems. We develop an algorithm for this class of semi-infinite transportation problems. The algorithm is a primal dual method which is a generalization of the classical algorithm for finite transportation problems. The most important aspect of our paper is that we can prove the convergence result for the algorithm. Finally, we implement some examples to illustrate our algorithm.

MSC: 90C34; 90C48

Keywords: Semi-infinite transportation problem; Algorithm; Optimal solution

1. Introduction

In this paper we mainly consider the algorithm for the continuous transportation problem (CTP). A large number of papers [1,2,4–11,13] have appeared in the literature on this problem. They have mostly been concerned with the duality theory of the CTP and the existence of optimal solutions for such a problem. Only a few of papers [1,2,10,11,13] discuss the algorithms for some special classes of CTPs.

The classical transportation problem (TP) is a linear program posed in R^{mn} as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i=1,2,\ldots,m,\\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j=1,2,\ldots,n,\\
& x_{ij} \geqslant 0, \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n.
\end{aligned}$$

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The decision variables x_{ij} represent the amount shipped from source i to destination j. The demand at destination j is b_j and the supply at source i is a_i. In order to make the problem consistent, we need $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. The CTP is a continuous version of the classical TP. Now we formulate the CTP as follows:

$$\begin{aligned} &\text{minimize} && \int_{X\times Y} c(x,y)\,\mathrm{d}\rho(x,y) \\ &\text{subject to} && \rho(S\times Y)=\mu_1(S) \quad \text{for any measurable set } S \text{ in } X, \\ &&& \rho(X\times S)=\mu_2(S) \quad \text{for any measurable set } S \text{ in } Y, \\ &&& \rho \geqslant 0. \end{aligned}$$

Here ρ, μ_1, and μ_2 are nonnegative regular Borel measures and c is a continuous function. X and Y are compact Hausdorff spaces with $\mu_1(X)=\mu_2(Y)$. There are numerous applications in the field of the CTP. For example, in Civil Engineering, we have the following problem: Let $\mu_1(x_i)$ be the amount of units of soil in n distinct locations $x_i (i = 1, \dots, n)$. We wish to transport the soil from the above n distinct locations to a certain highway which is under construction. The amount of soil that is needed for every given length E in the highway is $\mu_2(E) = \int_E h(y)\,\mathrm{d}y$, where $h(y)$ is a given function. The question is if the cost needed to move every unit amount of soil from location x_i to the yth position of the highway is $c(x_i, y)$, then what is the minimum cost that needed for the transportation of the soil? This TP is formulated by $X = \{x_1, \dots, x_n\}$ and Y, where Y is a closed interval in R and μ_1, μ_2 are defined as above with $\mu_1(X) = \mu_2(Y)$. In this paper we will develop an algorithm for solving this kind of problem. It is well known that the dual problem for the continuous transportation problem (DCTP) has the following form:

$$\begin{aligned} &\text{maximize} && \int_X r(x)\,\mathrm{d}\mu_1(x) + \int_Y s(y)\,\mathrm{d}\mu_2(y) \\ &\text{subject to} && r(x)+s(y)\leqslant c(x,y) \quad \text{for each } (x,y)\in X\times Y, \end{aligned}$$

where r and s are continuous functions on X and Y, respectively. From the duality theory of CTP, we know if ρ is feasible for CTP and (r, s) is feasible for DCTP, then $\int_{X\times Y} c(x,y)\,\mathrm{d}\rho(x,y) \geqslant \int_X r(x)\,\mathrm{d}\mu_1(x) + \int_Y s(y)\,\mathrm{d}\mu_2(y)$. We denote $v(\text{CTP})$ and $v(\text{DCTP})$ as the optimal values for CTP and DCTP, respectively. Kretschmer [7] has proved a strong duality result for CTP and DCTP. Essentially this states that $v(\text{CTP}) = v(\text{DCTP})$. Anderson and Philpott [2] developed an algorithm to solve a simple version of CTP with μ_1 and μ_2, which are both Lebesgue measures. Anderson and Nash [1] and Lewis [10] discussed the semi-finite TP, which has some relation to the version of our problem. From Wu [13], we know the CTP has an optimal solution at an extreme point of the feasible region. Anderson and Nash [1] proved that there is an optimal solution for DCTP.

In this paper we consider the case $X = \{x_1, \dots, x_n\}$, $Y = [0, 1]$, and μ_1 and μ_2 are defined as above. We intend to modify the algorithm in Anderson and Nash [1] and give a convergence proof for this algorithm to solve CTP. We characterize the extreme points structure for CTP and DCTP in Section 2. In Section 3 we use the idea from Anderson and Nash [1] to give an algorithm for CTP and prove the convergence result for this algorithm. Finally, we implement some numerical examples to illustrate our algorithm in Section 4.

2. Extreme points

In this section we discuss extreme points for CTP. From now we let $X = \{x_1, \dots, x_n\}$, $Y = [0, 1]$, μ_1 be a discrete measure concentrated on X, and μ_2 be an absolutely continuous measure with respect to Lebesgue measure.

From Wu [13], we have the following Theorems 2.1 and 2.2.

Theorem 2.1. *Let ρ be a regular Borel measure with* $\operatorname{supp}(\rho) = \bigcup_{i=1}^{n}\{x_i \times S_i\}$, *where S_i is a finite union of intervals of R for $i = 1, 2, \dots, n$ and the length of $S_i \cap S_j$ is equal to* 0 *for $i \neq j$. If ρ is feasible for CTP, then ρ is an extreme point of CTP.*

If ρ is feasible for CTP, then $\operatorname{supp}(\rho) \subseteq \bigcup_{i=1}^{n}\{x_i \times Y\}$. Now we assume that $\operatorname{supp}(\rho) = \bigcup_{i=1}^{n}\{x_i \times S_i\}$, where S_i is a finite union of intervals of R for $i = 1, \dots, n$ and $\rho(x_i \times E) = \int_E h(y)\,\mathrm{d}y$ for every measurable set $E \subset S_i$, where $\inf_{y\in S_i} h(y) > 0$ for $i = 1, 2, \dots, n$.