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On the equivalence of parametric contexts for linear inequality systems

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Abstract

In this paper we analyze the connections among different parametric settings in which the stability theory for linear inequality systems may be developed. Our discussion is focussed on the existence, or not, of an index set (possibly infinite). For some stability approaches it is not convenient to have a fixed set indexing the constraints. This is the case, for example, of discretization techniques viewed as approximation strategies (i.e., discretization regarded as data perturbation). The absence of a fixed index set is also a key point in the stability analysis of parametrized convex systems via standard linearization. In other frameworks the index set is very useful, for example if the constraints are perturbed one by one, even to measure the global perturbation size. This paper shows to what extent an index set may be introduced or removed in relation to stability.

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1. Introduction

This paper is concerned with different frameworks in which the stability of a given linear inequality system (with possibly infinitely many constraints) may be analyzed. Specifically, we deal with three different frameworks: normalized parametric context with *index set T*, parametric context with index set T, and general parametric context, abbreviated by $\mathrm{NP}(T)$, $\mathrm{P}(T)$ and GP, respectively.

In context $\mathrm{NP}(T)$ we consider a 'normalized' parametrization in the sense that it obeys to a certain pattern. Indeed, once the index set T is selected, both the space of parameters and the mapping assigning linear systems to parameters are prescribed. Formally, when we are confined to context $\mathrm{NP}(T)$, we consider the set Θ of all the systems in the form

$$\sigma := \{a_t' x \geqslant b_t,\ t \in T\}, \tag{1}$$

where T is an arbitrary, but fixed, set of indices, x and a_t belong to $\mathbb{R}^n$, $b_t \in \mathbb{R}$, and y' denotes the transpose of $y \in \mathbb{R}^n$. When T is infinite we are dealing with linear semi-infinite systems. The functions $t \mapsto a_t$ and $t \mapsto b_t$ have no particular

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property and, so, $\Theta \equiv (\mathbb{R}^n \times \mathbb{R})^{\mathrm{T}}$. Actually, in this context, $(\mathbb{R}^n \times \mathbb{R})^{\mathrm{T}}$ is the *parameter space* which is identified with the set of systems Θ. Moreover, a natural bijection is considered, which assigns to each parameter $\binom{a_t}{b_t}_{t\in T} \in (\mathbb{R}^n \times \mathbb{R})^{\mathrm{T}}$ the linear system $\{a_t' x \geqslant b_t,\ t \in T\}$. In this context, each coefficient may be independently perturbed, giving rise to a perturbed system with the same number (cardinality) of constraints. It follows the classical approach of Robinson [21] in the sense that *arbitrary* perturbations of *all* the coefficients are considered. The size of the perturbation is measured by the *extended distance* $\overline{d} : \Theta \times \Theta \to [0, +\infty]$ given by

$$\overline{d}(\sigma_1, \sigma) = \sup_{t\in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty, \tag{2}$$

where $\sigma_1 := \{(a_t^1)' x \geqslant b_t^1,\ t \in T\}$, and $\|\cdot\|_\infty$ denotes the Chebyshev norm in any finite-dimensional Euclidean space. In this way, Θ is endowed with the uniform convergence topology.

Roughly speaking, context NP(T) provides a theoretical setting for modeling those situations in which systems have coefficients whose values either are only approximately known or they have to be rounded off in the computing process. Therefore, we may be actually considering a different system, σ_1, proximal to the original one. The stability theory in this framework has been studied in several papers (e.g., [4,7,12,14,15]). The *continuous case*, i.e., when T is assumed to be a compact Hausdorff space and $\binom{a_t}{b_t}$ depends continuously on $t \in T$, has been analyzed in [2,9]. Some stability results for structurally richer contexts can also be traced out from the literature, for instance, by requiring differentiability assumptions; see, for instance, Jongen et al. [18], for the case of C^1 data.

A different parametric context arises by considering the family of systems in the form

$$\sigma_\theta := \{a_t(\theta)' x \geqslant b_t(\theta),\ t \in T\}, \tag{3}$$

where the parameter θ runs over an arbitrary metric space (Θ, d) and, for each $t \in T$, the function $\theta \mapsto \binom{a_t(\theta)}{b_t(\theta)}$ is continuous on Θ. We shall refer to this parametric context as P(T). This parametric approach may be used to describe models in which only *specific perturbations* of *specific coefficients* are allowed. In this framework perturbations fall on the parameter and, when T is infinite, a small perturbation of the parameter θ may cause a large perturbation in some coefficients of the original system. This fact provokes that the stability theory of the feasible set in context P(T) is notably different from the one in NP(T), as it is emphasized in [6]. This approach is followed by authors as Zlobec [24] in the finite case, i.e, T finite (see also [19] which includes the semi-infinite case), and Jongen and Stein [17], confined to the context of parametrized families of problems with C^1 data.

A special case of parametrized systems (3) comes from the field of the so-called *multiparametric programming*, where the parameter ranges on $\mathbb{R}^k$. In this framework, parametric programming techniques systematically subdivide the parameter space into characteristic regions where the optimal value and an optimizer are given as explicit functions of the parameters. As it is pointed out in Filippi [8], in recent years a new interest in multiparametric programming arose from so-called *model predictive control*, a well-known technique in the system theory and optimal control community. The pioneer work of Gal and Nedoma [11] introduces a general and systematic procedure to solve a multiparametric right-hand side linear programming problem. See Gal [10] for a vast bibliography on parametric linear programming, updated to the early 1990s.

Context GP is concerned with the case in which 'the whole system' may present perturbations: a perturbed system may have even a different amount of constraints. The situation may be modeled by including dependence on the parameter for the index set (see again [19,24]) or with no index set. In this last case each linear inequality system is directly identified with the subset of $\mathbb{R}^{n+1}$ formed by the coefficient vectors of the constraints. This broad scope context GP can be formalized through the concept of *mapping of parametrized systems*, defined as follows (see [6]):

Definition 1. Let Θ be an arbitrary metric space, which will be considered as the parameter space. We define a *mapping of parametrized systems* as a set-valued mapping $\sigma : \Theta \rightrightarrows \mathbb{R}^{n+1}$, assigning to each parameter θ a subset $\sigma(\theta)$ of $\mathbb{R}^{n+1}$, which can be identified with the linear inequality system

$$\sigma_\theta := \left\{ a' x \geqslant b,\ \begin{pmatrix} a \\ b \end{pmatrix} \in \sigma(\theta) \right\}.$$

Context GP becomes the natural framework for modeling different situations. At this moment we point out the case of parametrized convex inequality systems, when the 'standard linearization' (by means of subdifferentials) is considered.