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# Biorthogonal radial multiresolution in dimension three[☆]

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ABSTRACT

In this paper, we present the definition and the relative theorems of the biorthogonal radial multiresolution in dimension three. Unlike the orthogonal case, there exist real-valued dual radial scaling functions with compact support in the biorthogonal case. The associated Mallat algorithm can be simply performed in terms of classical biorthogonal filters.



## 1. Introduction

Based on classical multiresolution analysis (MRA) in Euclidean space $\mathbb{R}^n$, wavelets have been developed in many branches, such as orthogonal wavelets, biorthogonal wavelets, frames, multi-wavelets, field wavelets, wavelets packages and lifting, etc. The compact support, symmetry and high order vanishing moments of the wavelets are always expected to be good and important properties. However in the orthonormal case, there do not exist real-value compactly supported scaling and wavelet functions with symmetry or antisymmetry (excepted Haar basis) [2,4]. To recover the symmetry, many approaches are put forward, such as biorthogonal wavelets, m-band wavelets and vector wavelets, etc.

For the analysis of the radially symmetric functions, the key is to exploit this symmetry in the construction of the corresponding wavelet transforms in order to reduce the computational effort. The radial property would naturally occur when separating variables in polar coordinates and treating the spherical and radial parts separately. There is much literature dealing with multiresolution analysis on spheres, (see e.g. [1] and the references therein). The continuous radial wavelet analysis has been established based on the convolution structure of the radial functions instead of the usual translation in $\mathbb{R}^d$ (see e.g. [7,8,10]). Essentially the same concept is underlying the approach of Epperson and Frazier, where the radial wavelet expansions in $\mathbb{R}^d$ are constructed. Sampling lattices with spatial discretization are determined by the positive zeros of the related Type-I Bessel functions [5]. The spatial lattice is equidistant only in the special cases $d = 1$ and $d = 3$. There seems to be no general rigorous approach available for the construction of orthogonal radial wavelets in arbitrary dimensions, although radial multiresolution analysis has been considered in some special cases. In [9], Holger Rauhut and Margit Rösler construct radial multiresolution analysis and orthonormal radial wavelets in $\mathbb{R}^3$, which contains almost all the results of the classical MRA. Additionally they present a concise characterization of the radial scaling function in terms of

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the even classical scaling function on $\mathbb{R}$. In particular, it implies that there do not exist any real-valued compactly supported orthonormal radial scaling function similar to the classical case. In this paper, we improve the above result to biorthogonal radial multiresolution. In the biorthogonal case, it is possible to obtain real-valued dual radial scaling functions with compact support. Radial filters and the Mallat algorithm could be represented by the classical biorthogonal filters, which is not of concern in the orthogonal case in [9]. Furthermore as an example, the 9/7 biorthogonal radial wavelets with compact support are shown.

This paper is organized as follows. In Section 2, we recall radial multiresolution analysis in $\mathbb{R}^3$ and some notations. In Section 3, we present biorthogonal radial multiresolution analysis and establish its construction method from the classical biorthogonal MRA. For example, we give the biorthogonal 9/7 radial scaling function and wavelets. Finally, the Mallat decomposition and reconstruction algorithms are discussed, where radial filters are easily obtained from classical filters.

## 2. Radial multiresolution analysis

The group $SO(d)$ denotes the set of rotations of d-dimension space. Suppose $F \in L^2(\mathbb{R}^d)$ is radial, i.e. $F(Ax) = F(x)$ a.e. for all $A \in SO(d)$. Then there is a unique $f \in L^2(\mathbb{R}_+, \omega_{d/2-1})$ such that $F(x) = f(|x|)$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. For $\alpha \geq 0$, the measure $\omega_\alpha$ on $\mathbb{R}_+ = [0, \infty)$ is defined by

$$\mathrm{d}\omega_\alpha(r) = (2^\alpha \Gamma(\alpha+1))^{-1} r^{2\alpha+1}\mathrm{d}r.$$

It implies that $\|F\|_2 = \|f\|_{2,\omega_{d/2-1}}$, where $\|\cdot\|_2$ is taken with respect to the normalized Lebesgue measure $(2\pi)^{-d/2}dx$ on $\mathbb{R}^d$. For $f \in L^2(H_\alpha) := L^2(\mathbb{R}_+, \omega_\alpha)$, the Hankel transform is defined by

$$\widehat{f}^\alpha(\lambda) = \int_0^\infty j_\alpha(\lambda r) f(r)\mathrm{d}\omega_\alpha(r),$$

where the normalized Bessel function $j_\alpha(z)$ is

$$j_\alpha(z) = \Gamma(\alpha+1)(z/2)^{-\alpha} J_\alpha(z), \qquad J_\alpha(z) = \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n+\alpha}}{n!\Gamma(n+\alpha+1)}.$$

Plancherel Theorem of the Hankel transform holds, which states $f \mapsto \widehat{f}^\alpha$ is a self-inverse and isometric isomorphism of $L^2(H_\alpha)$. If $F \in L^2(\mathbb{R}^d)$ is radial with $F(x) = f(|x|)$, then its Fourier transform is also radial with $\mathcal{F}(F)(\xi) = \widehat{f}^{(d/2-1)}(|\xi|)$. It is due to the fact that

$$j_{d/2-1}(|z|) = \int_{S^{d-1}} \mathrm{e}^{-\mathrm{i}\langle z,\xi\rangle}\mathrm{d}\sigma(\xi),$$

where $\mathrm{d}\sigma$ denotes the spherical surface measure normalized according to $\mathrm{d}\sigma(S^{d-1}) = 1$. In contrast, the usual group translation $x \mapsto F(x+y), y \in \mathbb{R}^d$, will no longer be radial (apart from trivial cases). However, the translation on the spherical means

$$MT_r F(x) := \int_{S^{d-1}} F(x + r\xi)\mathrm{d}\sigma(\xi), \quad r \in \mathbb{R}_+.$$

The translation of $F$ are again radial. Moreover, $\|MT_r F\|_2 \leq \|F\|_2$. Thus $MT_r$ induces a norm-increasing linear mapping $T_r : L^2(H_{d/2-1}) \longrightarrow L^2(H_{d/2-1})$ defined by

$$T_r f(|x|) := MT_r F(x),$$

where $f$ and $F$ are defined as above. Let $\alpha = d/2 - 1$, then

$$T_r f(s) = C_\alpha \int_0^\pi f(\sqrt{r^2 + s^2 - 2rs\cos\varphi}) \sin^{2\alpha}\varphi \mathrm{d}\varphi, \tag{1}$$

with $C_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)}$. For $\alpha = d/2-1$ or general $\alpha \geq -1/2$, it defines a norm-decreasing generalized translation on $L^2(H_\alpha)$, which is different from the classical case (see [6]).

The dilation operator $D_a$ in $L^2(H_\alpha)$ is a linear operator defined by

$$D_a f(r) := \frac{1}{a^{\alpha+1}} f\left(\frac{r}{a}\right), \quad a > 0.$$

Radial multiresolution analysis in $\mathbb{R}^3$ associated with the Bessel-Kingman hypergroup $H_\alpha$ with $\alpha = 1/2$ has been discussed in [9]. In this paper, we continue to use the notations in [9], i.e.

$$\mathrm{d}\omega(r) := \mathrm{d}\omega_{1/2}(r) = \sqrt{\frac{2}{\pi}} r^2 \mathrm{d}r, \qquad \widehat{f} := \widehat{f}^{1/2}, \qquad j(r) := j_{1/2}(r) = \frac{\sin(r)}{r}.$$