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G^2 cubic transition between two circles with shape control

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Abstract

This paper describes a method for joining two circles with an S-shaped or with a broken back C-shaped transition curve, composed of at most two spiral segments. In highway and railway route design or car-like robot path planning, it is often desirable to have such a transition. It is shown that a single cubic curve can be used for blending or for a transition curve preserving G^2 continuity with local shape control parameter and more flexible constraints. Provision of the shape parameter and flexibility provide freedom to modify the shape in a stable manner which is an advantage over previous work by Meek, Walton, Sakai and Habib.

Keywords: G^2 transition; Path planning; Cubic Bézier; Computer-aided design (CAD)

1. Introduction

Fair path planning is one of the fundamental problems, with numerous applications in the fields of science, engineering and technology such as highway or railway designing, networks, robotics, GIS, navigation, CAD systems, collision detection and avoidance, animation, environmental design, communications and other disciplines. One of the main approaches to path planning is through the use of cubic spline functions.

Parametric cubic curves are popular in CAD applications because these are the lowest degree curves that allow inflection points (where curvature is zero). Such curves are suitable for the composition of G^2 blending curves. The Bézier form of a parametric cubic curve is mostly used in CAD and CAGD applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font design. A fair curve should have curvature extrema only where explicitly desired by the designer. The importance of using fair curves in the design process is well documented in the literature [1–5].

Consumer products such as ping-pong paddles can be designed by blending circles [6]. For applications in the design of highway or railway routes, or trajectories of mobile robots, it is desirable that the transitions be fair. In the discussion about geometric design standards in AASHO (American Association of State Highway Officials), Hickerson [7] (p. 17) states that "Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of *gradually increasing or decreasing radii* and, at the same

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time, introducing an appearance of forced alignment". The importance of this design feature which is highlighted in [8] links vehicle accidents to inconsistency in highway geometric design.

Cubic curves, although smoother, are not always helpful since they might have unwanted cusps, loops, up to two inflection points, and curvature extrema [2,9–12]. According to Farin [1], curvature extrema of a fair curve "should only occur where explicitly desired by the designer". B-splines and Bézier curves do not normally allow this. However, it can be accomplished when designing with spiral segments. A spiral is free of local curvature extrema, making spiral design an interesting mathematical problem with importance for both physical and aesthetic applications [13–20]. The clothoid, Cornu or non-polynomial cubic spiral has been used in highway designing and robot path planning for many years [21–24]. A major drawback in using this spiral is that the spiral segment currently used is neither polynomial nor rational. It is thus not easily incorporated in CAD/CAM/CAGD packages mostly based on NURBS (Non-Uniform Rational B Splines).

Walton and Meek [6] considered planar G^2 transition between two circles with a single fair cubic Bézier curve. They showed that there is no curvature extremum in the case of an S-shaped transition, and that there is a curvature extremum in the case of C-shaped transition. Use of a single curve rather than two segments has the benefit that designers have fewer entities to be concerned with. Habib and Sakai [25] simplified the analysis of Walton and Meek [6] and provided a less restrictive ratio of the larger to the smaller radii of the given circular arcs.

The objectives of this paper are to

- Further simplify and extend the analysis in [6,25].
- Obtain a fair G^2 cubic transition between two circles with more flexible constraints than in [6].
- Introduce a parameter to control the transition curve while preserving its important shape features.
- Visualize the relationship between shape control parameter and numerical value of the arc-length.

Provision of flexibility in the selection of radii of circular arcs and local shape control of the transition curve is certainly an advantage over previous work in [6,25].

2. Background

2.1. Notation and conventions

The usual Cartesian coordinate system is presumed. Bold face is used for points and vectors, e.g.,

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}.$$

The Euclidean norm or length of a vector $\boldsymbol{a}$ is denoted by the notation

$$\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2},$$

and $\boldsymbol{a} \parallel \boldsymbol{b}$ means the vector $\boldsymbol{a}$ is parallel to vector $\boldsymbol{b}$. The positive angle of a vector $\boldsymbol{a}$ is the counterclockwise angle from the vector $(1, 0)$ to $\boldsymbol{a}$. The derivative of a function f is denoted by f'. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the signed z-component of the usual three-dimensional cross-product of two vectors in the xy plane, e.g.,

$$\boldsymbol{a} \times \boldsymbol{b} = a_x b_y - a_y b_x = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin\theta,$$

where θ is the counterclockwise angle from $\boldsymbol{a}$ to $\boldsymbol{b}$. $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the usual inner product.

The signed curvature of a parametric curve $\boldsymbol{P}(t)$ in the plane is

$$\kappa(t) = \frac{\boldsymbol{P}'(t) \times \boldsymbol{P}''(t)}{\|\boldsymbol{P}'(t)\|^3}, \tag{2.1}$$

when $\|\boldsymbol{P}'(t)\|$ is non-zero. Positive curvature has the center of curvature on the left as one traverses the curve in the direction of increasing parameter. For non-zero curvature, the radius of curvature, positive by convention, is $1/|\kappa(t)|$.